The Black Hole Information Paradox
Assignment 6

Due Saturday, 27 April 2021

1. **Entanglement entropy density of an infinite belt**
   In the lecture, we worked out the entanglement entropy of a portion of the AdS_3 boundary. Now consider AdS_{d+1} written as

$$ds^2 = -dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2. \tag{1}$$

   Consider the infinite belt specified by $x_1 \in [-\frac{l}{2}, \frac{l}{2}]$ with $x_2, \ldots x_{d-1} \in [-\infty, \infty]$. Find the holographic prediction for the entanglement entropy density of this belt. (You will obtain a density and not a number because the extent of the belt is infinite in the transverse directions.)

2. **Endpoint of the Page curve**
   Consider the simple model of islands described in the lecture. Consider a bulk AdS_5 metric

$$ds^2 = \frac{1}{z^2}\left(-h(z)dt^2 + \frac{dz^2}{h(z)} + dy^2 + d\vec{x}^2\right), \tag{2}$$

   where $\vec{x}$ denotes the 2 transverse directions and

$$h(z) = 1 - \frac{z^4}{z_h^4}. \tag{3}$$

   The geometry is terminated by a brane at $y = 0$. This geometry has two asymptotic regions, and we are interested in computing the union of the entropy of the region $[y_0, \infty]$ at $t = t_0$ on one boundary with the region $[y_0, \infty]$ at $t = t_0$ on the other boundary

   Find the value of y_0 below which islands *always* dominate this entropy calculation i.e. find the lowest value of y_0 so that the area of the curve that starts at $z = 0, y = y_0$ and ends perpendicularly on the brane is smaller than the area of the curve that goes through the horizon even at $t_0 = 0$.

   Your answer should be of the form $y_0 = kz_h$ and you need to find the number k. You may need to do some simple numerical work to find this value.

3. **Gravitating baths**
   Consider the black string geometry

$$ds^2 = \frac{1}{u^2 \sin^2 \mu}\left[-h(u)dt^2 + \frac{du^2}{h(u)} + d\vec{x}^2 + u^2 d\mu^2\right], \tag{4}$$

where $h(u) = 1 - \frac{u^{d-1}}{u_h^{d-1}}$. Here μ is an angular coordinate, and we terminate the geometry by branes placed at $\mu = \theta_1$ and $\mu = \pi - \theta_2$. This geometry provides a model of a *gravitating bath.*

Consider a minimal surface that runs from one one brane to the other.

(a) Show that if the surface is minimal with respect to its end-points it must satisfy

$$\frac{du}{d\mu} = 0, \tag{5}$$

at both endpoints.

(b) Show that $u = u_h$ (the horizon) also satisfies the bulk Euler-Lagrange equations for a minimal surface.

In fact, it turns out that $u = u_h$ is the only minimal surface that connects the branes. This provides evidence that if one asks a *gravitational question*, the Page curve is trivial. (However, even in gravitating systems, one can find appropriate nongravitational questions whose answer is given by a Page curve.)