17 Feb 2020

Lecture 11: the Hilbert space in global AdS

The metric of global AdS is given by

$$ds^2 = -(1+r^2)\,dt^2 + \frac{dr^2}{1+r^2} + r^2\, d\Omega_{d-1}^2$$

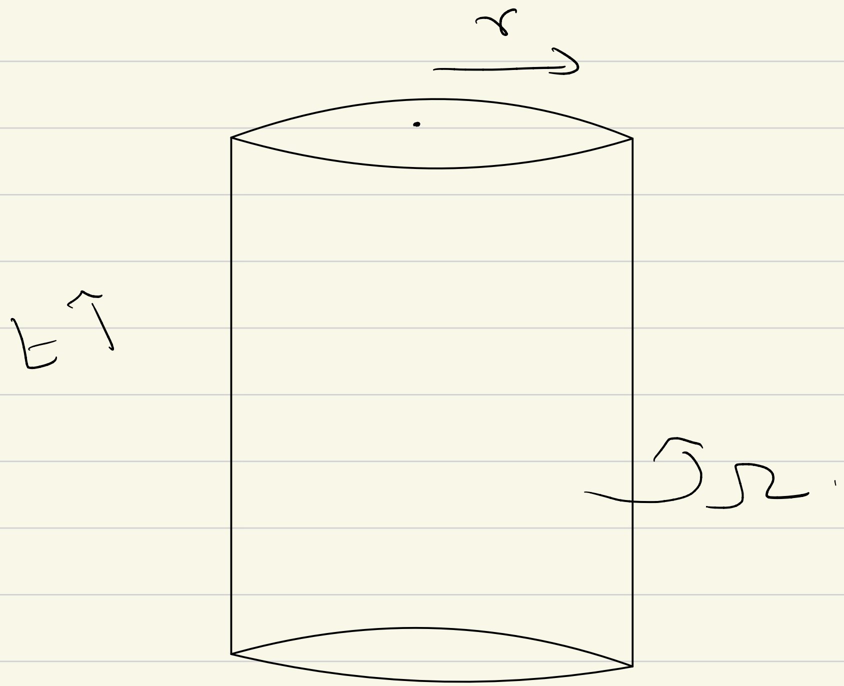
Here the AdS radius is set to

$$l_{AdS} = 1$$

Note we are considering

$$AdS_{d+1}$$

Empty global AdS can be pictured as a solid cylinder



We will later be interested in theories with gravity, but gravity will always be weak. A natural parameter is:

$$N = \frac{l_{ads}}{l_{planck}}$$

and we will always consider

$$N \gg 1$$

and N the largest $O(1)$ number around.

Let us start by discussing a free minimally coupled sclar field. This is a field that obeys

$$(\Box - m^2)\,\phi(t, r, \Omega) = 0$$

Before we look for solutions, we need to decide what kind of boundary conditions to impose.

If you are familiar with Euclidean versions of AdS/CFT, you might be familiar with settings where we set some nontrivial bdry conditions on ϕ as $r \to \infty$

But this corresponds to **deforming** the theory with a boundary source.

We are interested in the **autonomous** dynamics of the theory.

[Explain in lecture]

So we impose

$$\phi \to 0 \quad \text{as} \quad r \to \infty.$$

With this boundary condition we can immediately separate variables and find the following solutions

$$\phi(t, r, \Omega) = \sum_{n \geq 0} C_{n,l}\, a_{n,l}\, e^{-i(2n+l+\Delta)t}\, Y_l(\Omega)\, \chi_{n,l}(r) + h.c.$$

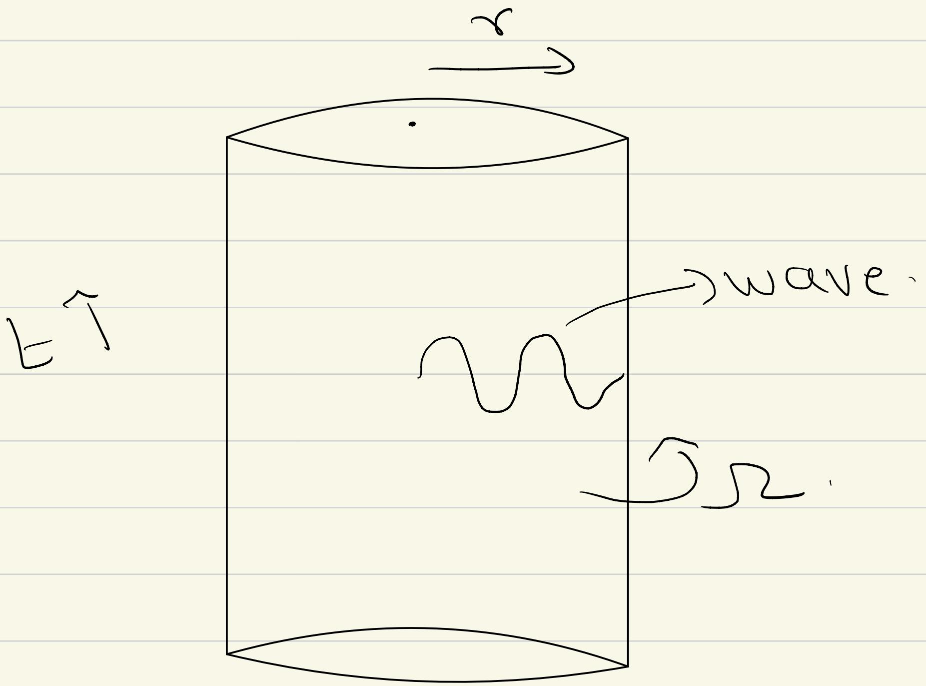
$$\chi_{n,l}(r) = r^l (r^2+1)^{-\frac{\Delta+2n+l}{2}}\, {}_2F_1\left(-n, -\Delta-n+\frac{d}{2}; \frac{d}{2}+l, -r^2\right)$$

and $C_{n,l}$ is a normalization factor found in the review [Appendix B]

Note something interesting. The frequencies are quantized!

$$\omega_n = 2n + l + \Delta.$$

This is a feature of the fact that global AdS is a box.



so for waves to fit an integral number of times, the frequencies must take on special values.

The operators $a_{n,l}$ are normalized so that

$$[a_{n,l}, a^{\dagger}_{n',l'}] = \delta_{nn'}\delta_{ll'}$$

Let us briefly make contact with AdS/CFT although we will largely not need to invoke it.

Even though we imposed $\phi \to 0$ we find that as $r \to \infty$

$$\phi \to \frac{1}{r^{\Delta}}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$$

We can define

$$\mathcal{O}(t,\Omega) = \lim_{r\to\infty} r^{\Delta}\phi(t,r,\Omega)$$

This picks up the asymptotic "tail" of the operator.

Then

$$\mathcal{O}_{n,l} = \int d\Omega \int_0^{\pi} dt\, \mathcal{O}(t,\Omega)\, e^{i(2n+l+\Delta)t}\, Y_l^*(\Omega)$$

are the modes of the "generalized free field" in the CFT dual to the bulk field.

Note

$$\mathcal{O}_{n,l} = \sqrt{G_{n,l}}\, a_{n,l}$$

$$\sqrt{G_{n,l}} = \frac{\Gamma(l+n+\Delta)}{\Gamma(n+1)\,\Gamma\left(-\frac{d}{2}+\Delta+1\right)\Gamma\left(\frac{d}{2}+l\right)}$$

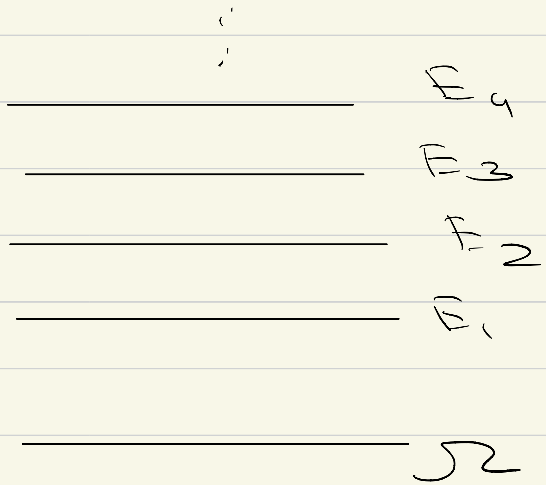
Now we turn to the Hilbert space. As usual, the perturbative vacuum is defined by

$$a_{n,l}|0\rangle = 0$$

Then we can construct a Hilbert space through

$$a^{\dagger}_{n_1,l_1}\, a^{\dagger}_{n_2,l_2} \cdots a^{\dagger}_{n_q,l_q}|0\rangle$$

Schematically, this looks like this



We can turn on interactions and then the precise eigenvalues shift and mix.

But the qualitative picture above persists.

These energy levels are dual to operator dimensions in the CFT, which remain discrete

Now, we note the following. We defined the Hilbert space by acting with modes $a^\dagger_{n,l}$

But we could also consider the action of

$$\int O(t_1, \Omega_1)\, e^{-i(2n+l+\Delta)t}\, Y_l^*(\Omega)\, dt\, d\Omega\, |0\rangle$$

This would produce a state parallel to

$$a^\dagger_{n,l} |\Omega\rangle$$

We can produce higher states in the Fock space as well:

$$\int dt_i\, d\Omega_i \Big[O(t_1,\Omega_1)\, e^{-i(2n_1+\Delta+l_1)t_1}\, Y^*_{l_1}(\Omega_1)$$

$$O(t_2,\Omega_2)\, e^{-i(2n_2+\Delta+l_2)t_2}\, Y^*_{l_2}(\Omega_2)$$

$$\ldots\ldots\ldots\ldots$$

$$O(t_q,\Omega_q)\, e^{-i(2n_q+\Delta+l_q)t_q}\, Y^*_{l_q}(\Omega_q)\Big] |\Omega\rangle$$

In general the Hilbert space is spamed by

$$O(f_1)\, O(f_2) \ldots O(f_q)\, |\Omega\rangle$$

where f_i are smearing functions and O

are boundary values of bulk fields.

Interacting Theory

This description of the Hilbert space is very useful because it generalizes to

1) the interacting theory
2) gravity.

Say the Hamiltonian of the full theory is H and the vacuum satisfies

$$H|0\rangle = 0$$

{It could be $H|0\rangle = E_0|0\rangle$ but then we just shif H by E_0}

Consider the set of operators

$$O(t_1) \ldots\ldots O(t_n)\,|0\rangle$$

Dropping Ω_i indices to lighten the notation going forward.

Under time-evolution

$$e^{-iH\tau}\, O(t_1) \dots O(t_n) |0\rangle$$

$$= e^{-iH\tau}\, O(t_1)\, e^{iH\tau}\, e^{-iH\tau}\, O(t_2)\, e^{iH\tau} \dots e^{-iH\tau}\, O(t_n)\, e^{iH\tau}\, e^{-iH\tau} |0\rangle$$

Just inserted 1   1   1.

$$= O(t_1 - \tau)\, O(t_2 - \tau) \dots\dots O(t_n - \tau) |0\rangle.$$

$$= O(t_1')\, O(t_2') \dots O(t_n') |0\rangle.$$

So a state of the form above evolves to another state of the same form under time evolution.

Result

The space spanned by

$$\text{span}\,\{\,O(f_1)\ldots O(f_n)\,|0\rangle\,\}$$

forms a superselection sector. No state of this form can evolve to a state outside this space

This is true **even in a theory of gravity**

In a theory of gravity, we hold **asymptotic boundary conditions** fixed and allow the bulk to vary arbitrarily.

But the notion of asymptotic operators makes sense even in gravity!

So even in a theory of gravity, the space formed by exciting the vacuum with asymptotic operators at arbitrary times forms a superselection sector

1) Note that the space is overcomplete. Even in perturbation theory

$$\int O(t)\left(e^{-i(2n+\Delta+l)t} a + b\, e^{i(2m+\Delta+l)t}\right)|0\rangle$$

is independent of b. (the second term picks up an annihilation operator.)

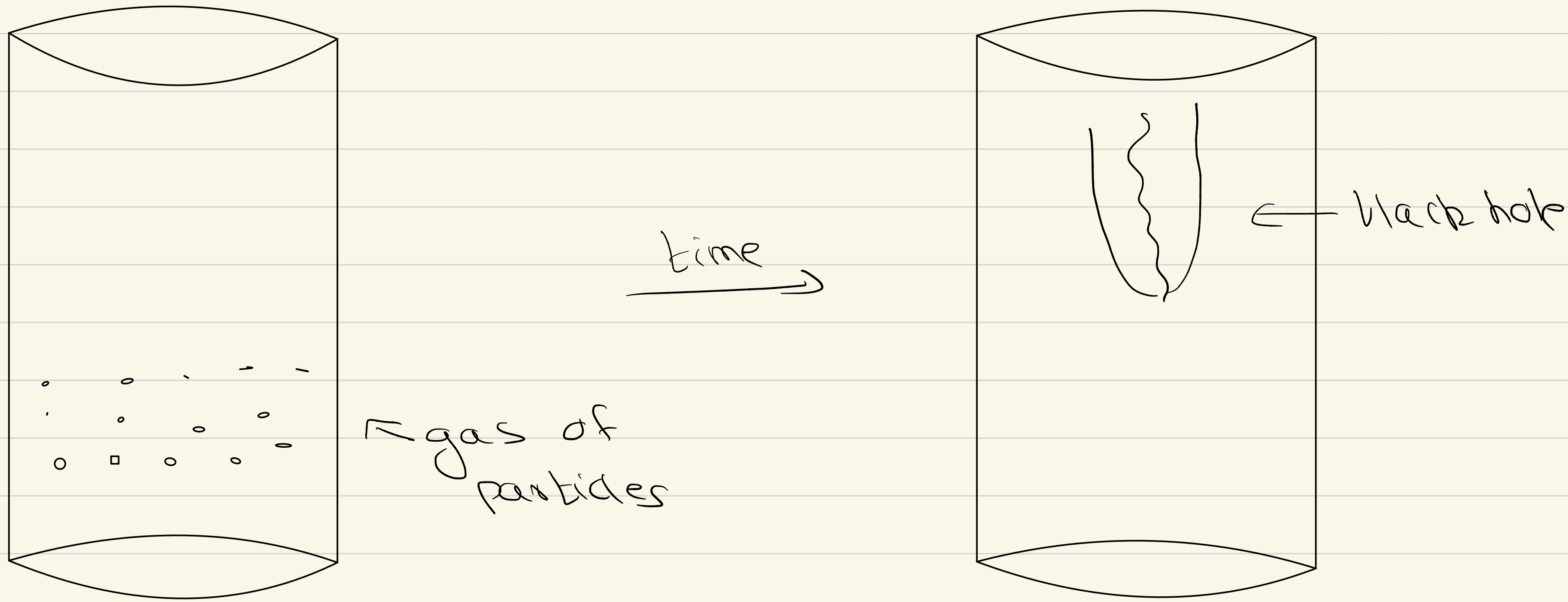
2) There is some magic happening here. In a theory of gravity, we expect that we should have strings & black holes & branes. How can we claim that a dynamically closed sector is obtained through

$$O(f_1) \ldots O(f_n)|0\rangle$$

where O are quantum field operators and f are arbitrary **space-time** functions.

Note

The point is that if we start with an arbitrary gas of particles and also allow ourselves to insert particles at later times, then we produce all states.



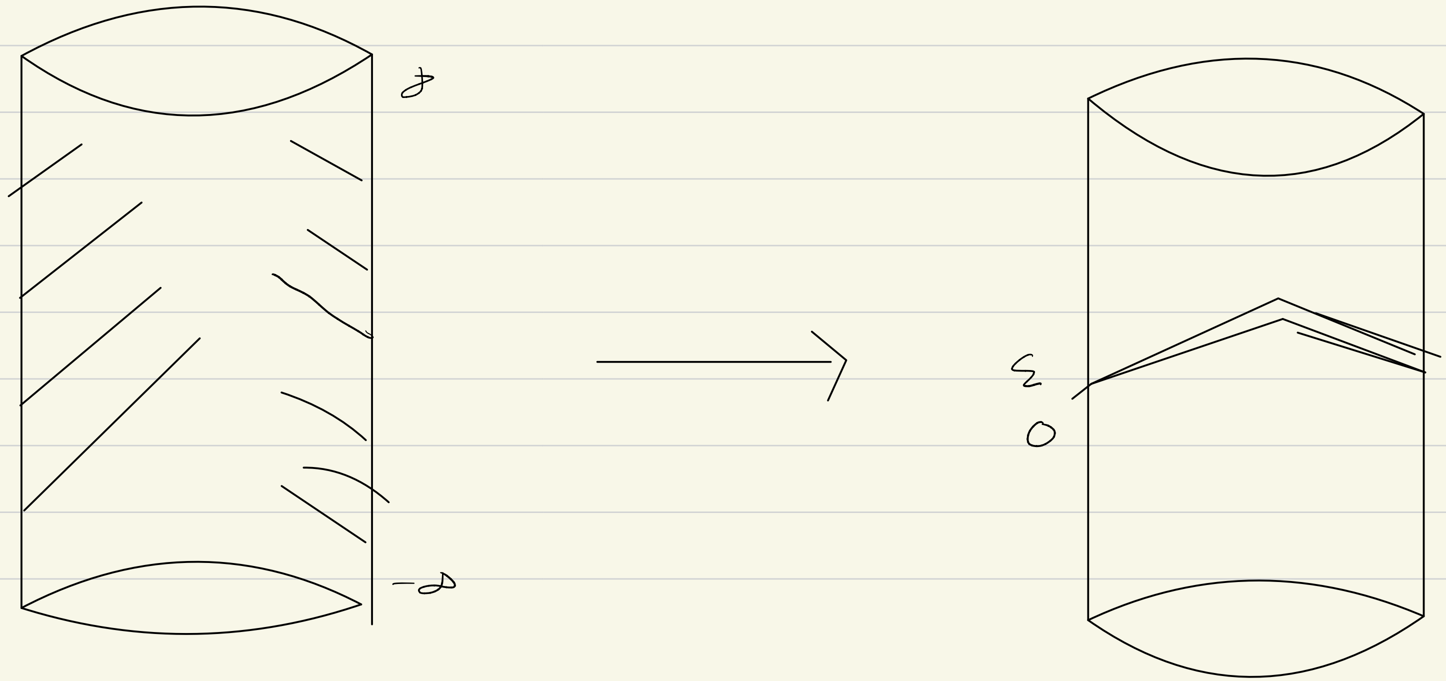
This is physically reasonable but we have proved this statement.

Squeezing the space

We will now prove a surprising result. The same space can be produced by

$$O(\tilde{f}_1) \ldots O(\tilde{f}_n) |0\rangle$$

where $\tilde{f}_1 \ldots \tilde{f}_n$ are functions with support only in the time-band $[0, \varepsilon]$



EMPHASIZE: THIS IS NOT THE PRINCIPLE OF HOLOGRAPHY OF INFORMATION!

THE PREVIOUS RESULT HOLDS ALSO IN QFTS AND DOESN'T INVOLVE GRAVITY.

Lets first consider one-particle states.
Let $|\psi\rangle$ be of the form

$$|\psi\rangle = O(f)|0\rangle$$ f has support anywhere in $(-\infty, \infty)$

Consider the set of states $O(\tau)|0\rangle$, for τ in $\{0, \varepsilon\}$

Say that $|\psi\rangle$ is orthogonal to all such states.

Then

$$\langle \psi | O(\tau) | 0 \rangle = 0, \qquad \forall \tau \quad \text{in } [0, \varepsilon]$$

But

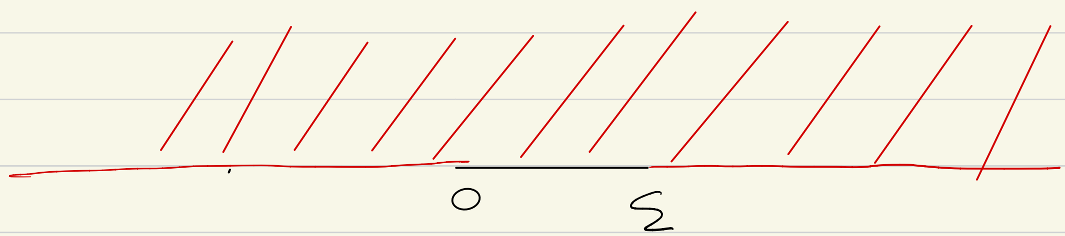
$$\langle \psi | O(\tau) | 0 \rangle = \sum_E \langle \psi | E \rangle \langle E | O(\tau) | 0 \rangle.$$

inserting a set of energy-eigenstates

$$= \sum_E \langle \psi | E \rangle \langle E | O(0) | 0 \rangle \, e^{iE\tau}$$

But this correlator is analytic when t is extended in the upper half plane.

But if we have such an analytic function



if it vanishes in $[0, \varepsilon]$ it must vanish on the entire real line.

but this is not possible since $|\psi\rangle$ is a linear combination of $O(t)|0\rangle$, so it must have

$$\langle \psi | O(t) | 0 \rangle \neq 0$$

for some t.

A similar argument works for states created with more operators

Let $|\psi\rangle$ be a state of the form

$$O(t_1)\, O(t_2) \, \ldots \, O(t_n)\, |0\rangle$$

for t_i arbitrary.

We will show it can be written as a <u>linear</u> <u>combination</u> of states of the form

$$O(\tau_1) \ldots O(\tau_n)|0\rangle$$

Proof is again by contradiction

Say $|\psi\rangle$ was orthogonal to all such states. Then

$$\langle\psi| O(\tau_1) \ldots O(\tau_n) |0\rangle = 0$$
$$\forall \ \tau_i \text{ in } [0, \varepsilon]$$

Now insert energy eigenstates

$$\langle\psi|E_1\rangle\langle E_1| O(\tau_1) |E_2\rangle\langle E_2|O(\tau_2) \ldots\ldots |E_n\rangle\langle E_n|O(\tau_n)|0\rangle$$

$$= e^{iE_1\tau_1} e^{iE_2(\tau_2-\tau_1)} \ldots\ldots e^{iE_n(\tau_n-\tau_{n-1})}$$
$$\langle\psi|E_1\rangle\langle E_1|O(0)|E_2\rangle \ldots\ldots \langle E_n|O(0)|0\rangle$$

Change variables to

$$z_1 = t_1$$
$$z_2 = t_2 - t_1$$
$$\cdots\cdots$$
$$z_n = t_n - t_{n-1}$$

Then

$$\text{previous expr} = e^{i \sum E_i z_i} \langle \psi | E_1 \rangle \langle E_1 | O(0) | E_2 \rangle \cdots \langle E_n | O(0) | 0 \rangle$$

This is analytic when z_i are extended in the upper half plane since E_i are positive

So if it vanishes whenever $t_i \in [0, \varepsilon]$ we will again find that it vanishes for all real t_i [edge-of-the-wedge theorem.]

Comments

1) This is a trivial example of the so-called Reeh Schlieder thm. In Minkowski QFT, this tells us that the Hilbert Space produced by

$$\phi(\tilde{f}_1) \dots \phi(\tilde{f}_n)|0\rangle$$

where $\tilde{f}_i$ are confined to some open region, is dense in the full Hilbert space

2) It is somewhat trivial because we are looking at states produced from **entire boundary** and our restriction is only that $\tau \in [0, \varepsilon]$. So we only need **positivity of energy** to prove the result.

[Otherwise we would need a "spectrum condition"]

3) Our result applies to theories of gravity

a) This is because asymptotic operators make sense even when the bulk is gravitational

b) we only used the positivity of energy

c) So the result that

"The Hilbert space generated by the action of asymptotic operators in a small time band $[0, \varepsilon]$ is dense in the full Hilbert space."

holds both for QFT and gravity in AdS.