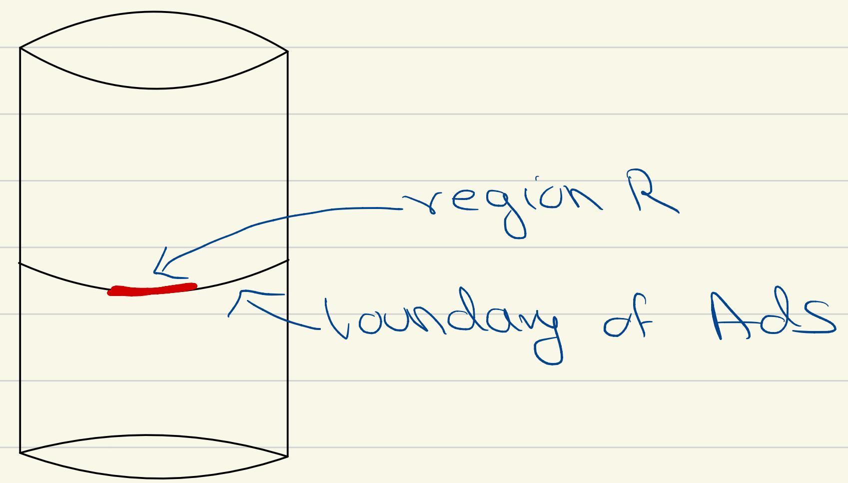
17 March 2021

Lecture 17: Entanglement entropy in AdS/CFT

As a prelude to the island formula we first review progress on understanding entanglement entropy in AdS/CFT.

These ideas form the basis of the island proposals.

The problem is as follows



Consider a CFT with a holographic dual

The Figure shows the CFT "living" on the boundary of AdS, but we can just think of it independently.

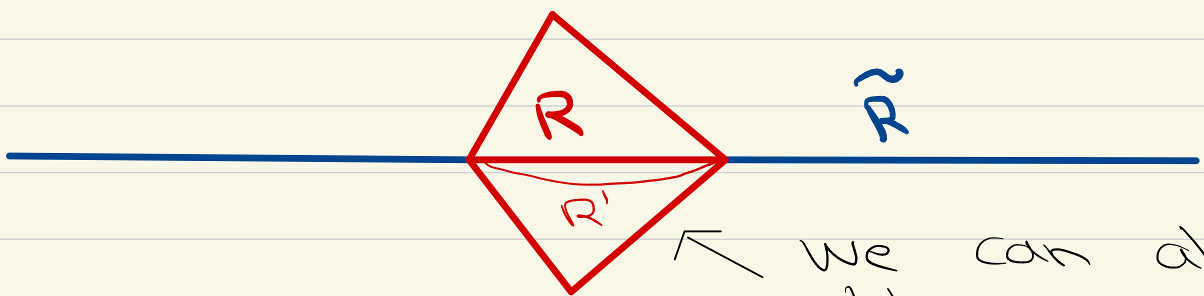
Consider a <u>spatial</u> <u>region</u> R of a Cauchy slice of the spacetime on which the CFT lives.

We want to find the entanglement entropy between R and $\tilde{R}$.

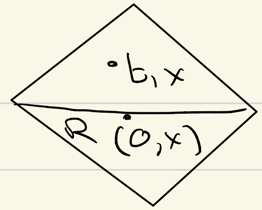
Note that this is an **entirely nongravitational** question.

Previously we have considered QFT results that are valid at energies smaller than the Planck scale. Here, we are asking a purely CFT question, so gravity does not exist at all!

Note that we can consider the causal diamond corresponding to R. any other slicing of this diamond (R') should give same E.E.



We can always consider the causal diamond that contains R [Explain]



Explanation

The operator algebra on the diamond is the same as the operator algebra on its base in a LQFT

We can write any operator as

$$O(t,x) = e^{iHt}\, O(0,x)\, e^{-iHt}$$

In LQFT a) $H = \int \mathcal{H}(x)\,dx$

b) Only $\int_R \mathcal{H}(x)\,dx$ is relevant since rest of integral has no effect

so

$$O(t,x) = \underbrace{e^{i\int_R \mathcal{H}(x)dx}\, O(0,x)\, e^{-i\int_R \mathcal{H}(x)\,dx}}_{\text{manifestly in the algebra of } R}$$

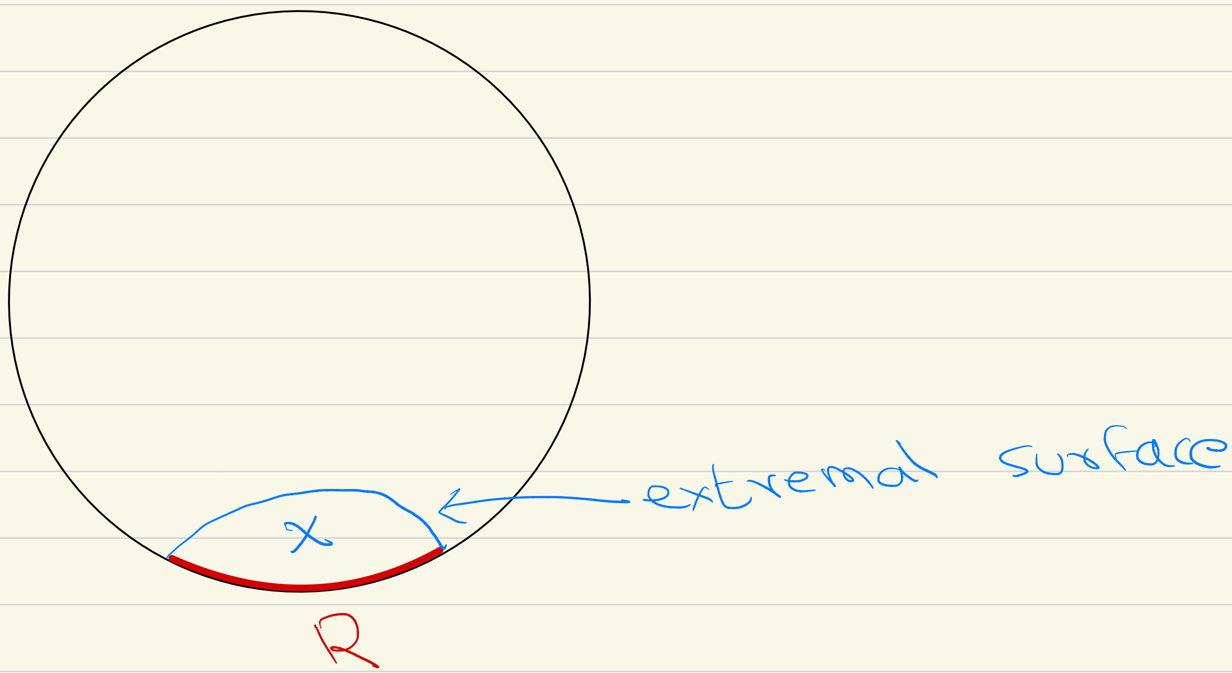
In theories with a gravitational dual, the proposal is

$$S(R) = \min\left[\mathrm{ext}\left[\frac{A}{4G} + S_{bulk}(X)\right]\right]$$

where the quantities are defined as follows

1) A is the area of a bulk surface anchored at R.

2) X is the region between the surface and R.
S_{bulk} is the free-field entropy of fields on X.



3) We are instructed to consider extremal surfaces
This involves "maximizing in time" and
"minimizing in space" [otherwise we could
lower area by moving the surface in time]

4) We are instructed to take the minimum
value of all such answers.

5) Note both sides are UV-divergent.

We can match UV-divergences on the two sides or compare entropies of two states.

6) The homology constraint states that X should have no boundaries except for R and the extremal surface

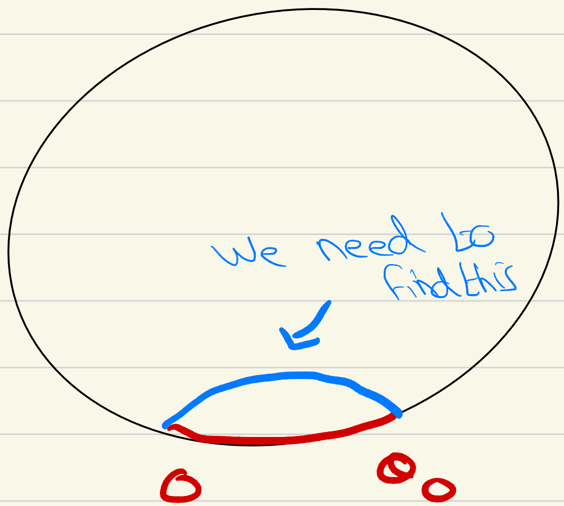
7) The subregion duality proposal states that:
"the CFT region R has information about the bulk region X."

Let us consider all these points in an example.

Consider global AdS_3

$$ds^2 = -(1+r^2)\,dt^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2$$

We will just consider the leading $\frac{A}{4G_N}$ term and consider the entropy of an interval $[0, \theta_0]$



From symmetry considerations we expect that this extremal surface will not move in t at all.

So we are only looking for a curve in $r-\theta$ space.

[Note of caution: Area $\neq$ geodesic length!]

The "area" we need to minimize is

$$\int \sqrt{\left(\frac{dr}{dz}\right)^2 \frac{1}{1+r^2} + r^2 \left(\frac{d\theta}{dz}\right)^2} \, dz.$$

For getting its equation, we can also consider

$$\int \sqrt{\frac{1}{1+r^2} + r^2 \left(\frac{d\theta}{dr}\right)^2} \, dr$$

We have a symmetry under $\theta \to \theta + \text{const}$ so we immediately get

$$\dot{\theta} = \frac{\pm c/r}{\sqrt{(r^2-c^2)(1+r^2)}}$$

Lets choose our surfaces so that initially $\dot{\theta}$ is positive [r is decreasing, so we want the surface to move "left"]

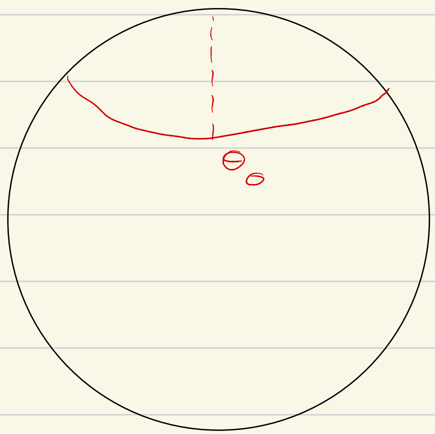
Then r decreases to a minimum value of "c". After that r increases and so we chose $\dot{\theta} < 0$.

So we can write

$$\theta = \theta_0 + \cot^{-1}\left[c\sqrt{\frac{1+r^2}{r^2-c^2}}\right] \quad \leftarrow \text{Ingoing branch}$$

$$\theta = \theta_0 - \cot^{-1}\left[c\sqrt{\frac{1+r^2}{r^2-c^2}}\right] \quad \leftarrow \text{return branch}$$

$$\delta\theta = 2\cot^{-1} c$$



To find the area we need to compute

$$A = 2 \int_{c}^{1/2\varepsilon} \frac{r\,dr}{\sqrt{(1+r^2)(r^2-c^2)}}$$

a) The factor of 2 arises due to the 2 segments "inbound" and "return"

b) The area diverges near $r=\infty$. So we have put a cutoff at $r=\frac{1}{2\varepsilon}$

Factor of 2 for convenience

then

$$A = -\log(1+c^2) + 2\log\frac{1}{\varepsilon}$$

But recall $c = \cot\frac{\delta\theta}{2}$

Eliminating c

$$A = 2\left(\log\frac{1}{\varepsilon} - \log\left(1+\cot^2\frac{\delta\theta}{2}\right)\right)$$

$$= 2\log\left(\sin\frac{\delta\theta}{2}\ \frac{1}{\varepsilon}\right)$$

$$\frac{A}{4G} = \frac{1}{2G} \log \frac{1}{\varepsilon} \sin \frac{\delta\theta}{2}$$

Using $c_N \leftarrow$ central charge $= \dfrac{3}{2G}$

this leads to

$$S = \frac{c_N}{3} \log \frac{1}{\varepsilon} \sin \frac{\delta\theta}{2}$$

We can <u>independently</u> show that in a 1+1 D CFT

$$S = \frac{c_N}{3} \log \frac{1}{\varepsilon} \sin \frac{\delta\theta}{2}$$

which matches exactly.

The UV divergence arises due to the short distance entanglement that we discussed previously

Despite its appearance in the formula,

note there are cutoff-independent quantities one can extract such as

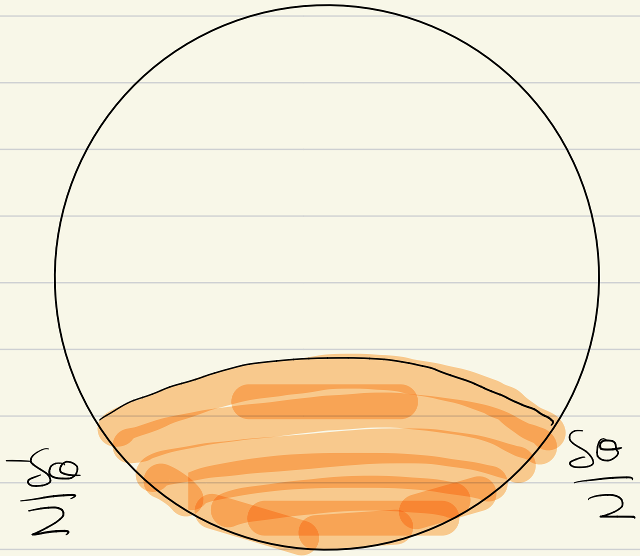
$$\frac{dS(\delta\theta)}{d\delta\theta}$$

so the formula is meaningful.

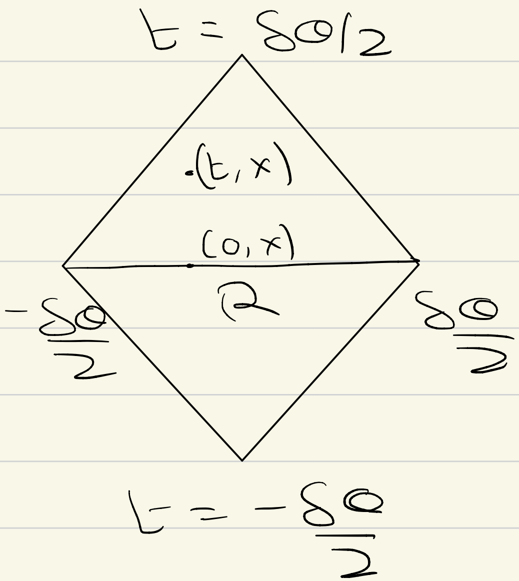
[In higher dimensions, the structure of UV divergences is more complicated.]

Let us now discuss the entanglement wedge.

The proposal is that the shaded bulk region is described by the boundary interval from $-\frac{\delta\sigma}{2}$ to $\frac{\delta\sigma}{2}$



On the boundary, we can think of information in a causal diamond



[See previous explanation]

"There is a timelike trajectory connecting every point in the shaded region to a point on the diamond."

Check:

Consider point at $\theta = 0$, $t = 0$ and $r = \cot \frac{\delta\theta}{2}$ in the WIR.

A light ray moves along

$$dt = \frac{dr}{(1+r^2)}$$

and so takes time

$$\cot^{-1} r_0$$

to reach from $r = r_0$ to $r = \infty$

So a ray from the $(r = \cot\frac{\delta\theta}{2}, \theta = 0, t = 0)$ reaches the boundary at $(\theta = 0, t = \frac{\delta\theta}{2})$

— tip of the diamond!

This does not "prove" the subregion duality but it tells us that we might use simple PDE techniques to map bdry diamond to bulk region.

The idea then is as follows. Consider some bulk field, ϕ. Say it obeys

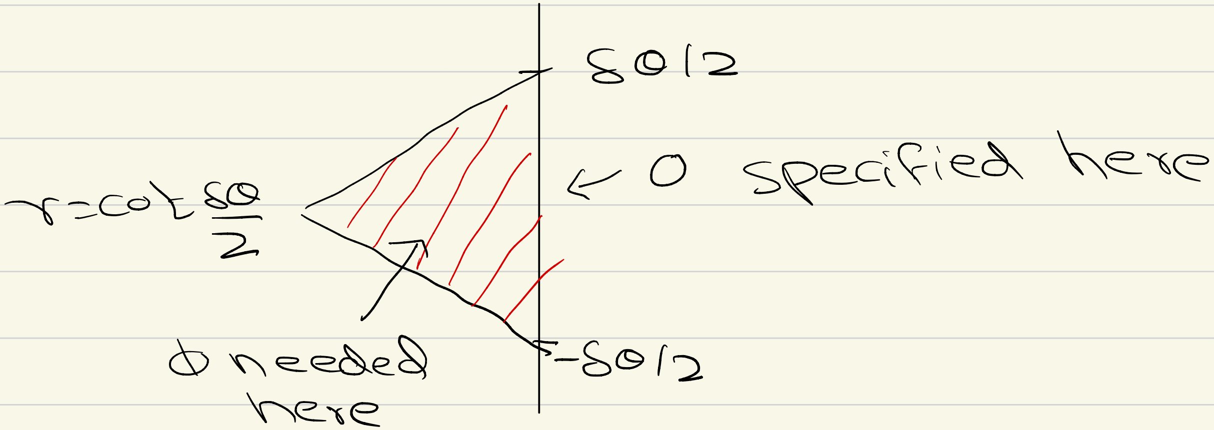
$$(\Box - m^2)\phi = 0. \qquad (1)$$

and is dual to an operator $\mathcal{O}$.

Then we solve (1), subject to boundary conditions

$$\lim_{r\to\infty} r^{\Delta} \phi(r,\theta,t) = \mathcal{O}(\theta,t),$$

$$\theta, t \in \text{bdry diamond}$$



This allows us to write

$$\phi(r, \theta', t') \quad \int O(\ , t)\, K(\theta, \theta', t, t', r) \underbrace{d\theta\, dt}_{\text{bdry diamond}}$$

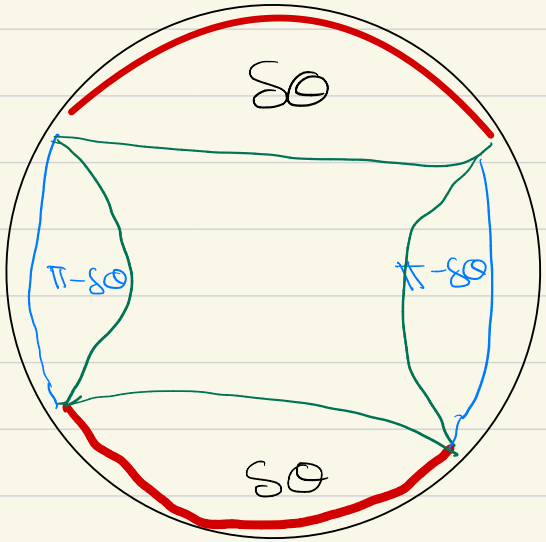
where the kernel "K" can be written explicitly.

[at least in "momentum space!"]

So, in this case we can **independently understand** why the bdry diamond matches the bulk region if we assume AdS/CFT.

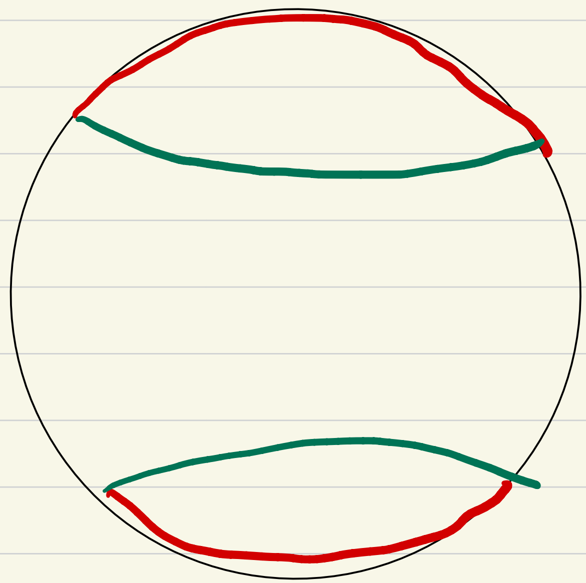
But it is not difficult to find examples of nontrivial entanglement wedges

Lets consider two intervals



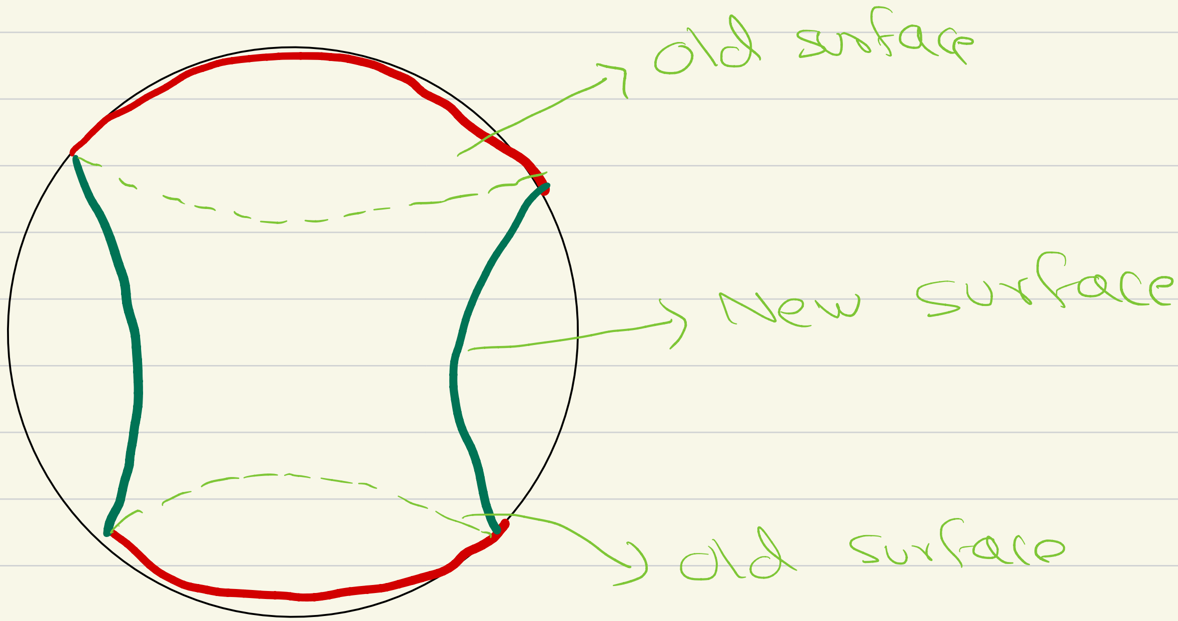
Here we have taken them of the same size and marked the complements in blue. RT surfaces in green.

For $\delta_0 < \frac{\pi}{2}$, the entropy is given by the sum of the previous two RT surfaces



But as the intervals grow, something remarkable happens.

For $\delta\theta > \frac{\pi}{2}$



the minimal surface becomes the one above.

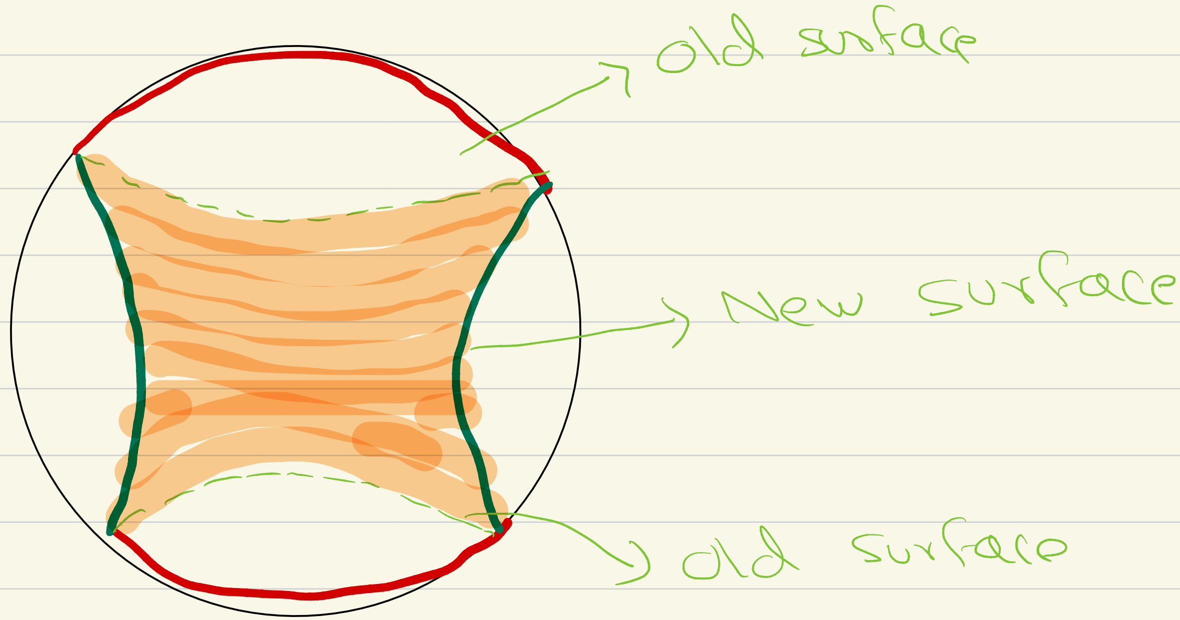
This is the out come of the principle instructing us to choose the minimum of extremal possibilities.

It was guaranteed that something like this had to happen since

$$S(R) = S(\tilde{R})$$

So when we consider disjoint intervals this phenomena **must** happen either for R or for its complement.

The subregion duality proposal now suggests something remarkable.



The shaded region is **out of causal contact** with the red region but the red region still has information about it.

This means that we **cannot** have a simple formula as previously

$$\phi_{bulk}(r, \theta', t') \neq \int_R O(x,t)\, K(\theta, \theta', t, t', r)\, d\theta\, dt$$

This is because we know that we have

$$[\phi_{bulk}(0), O(R)] \sim \frac{1}{N}$$

middle of AdS

non-zero due to possible gravitational effects.

by bulk causality.

[Explain]