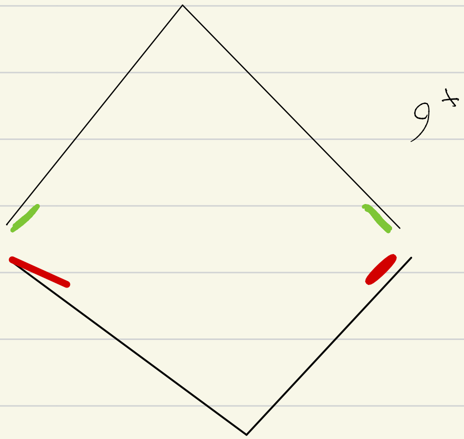
4 March 2020

Lecture 16

Entanglement entropy at $\mathcal{I}^+$

Before resuming our discussion of the monogamy paradox, clarify what we have shown.



If we **assume** (a) that the full UV theory shares low-energy properties of the low-energy theory i.e. that

$$T_{s,s'} = |\{s\}\rangle\langle\{s'\}| \in A_{-\infty}$$

and we assume a positive and real spectrum for the Hamiltonian of the full UV theory then:

all information about the state is available at $\mathcal{I}^+_-$

all information about the state is also available at $\mathcal{I}^-_+$ [future boundary of past null infinity.]

What we have **not** shown

a) full UV theory is well-defined

b) information at $\mathcal{I}^-_+$ is same as information at $\mathcal{I}^+_-$.

For instance, yesterday we discussed global symmetries [in low-energy tests.]

It may happen that such theories are in the "swampland."

Our discussion does not tell us whether or not this is the case.

Almost **all** recent discussions [at least in the string theory literature] take this perspective

"We assume there is a consistent UV-complete unitary theory that obeys Q.M. and then **explain** puzzles/paradoxes about black holes."

These discussions are interesting because they provide us with interesting **broader physical lessons** about quantum gravity.

eg. the principle of holography of information is independently interesting.

Even independent of black holes, it tells us that quantum gravity localizes information in surprising ways.

This is also easier to check experimentally than unitarity of black hole evaporation!

Monogamy paradox

Yesterday we discussed how, if one insists that the Hilbert space factorizes, one can construct a monogamy paradox even in empty AdS.

So this "toy paradox" tells us that if we insist that information in gravity must be localized like LQFTs, we run into paradoxes even in empty space.

We can explicitly construct a monogamy paradox about black holes by making the **same mistake.**

Let $|\Psi\rangle$ be a black hole state.

For concreteness consider a small black hole in AdS

Then we can use our construction to find operators A, B near the horizon so that

$$\langle\Psi| C_{AB} |\Psi\rangle > 2$$

Now in this case we can again find operators near infinity so that

$$Q|0\rangle = |\Psi\rangle \qquad Q_{B_2}|0\rangle = |B_2\rangle$$

$$Q_{B_1}|0\rangle = |B_1\rangle \qquad [\, |B_i\rangle = B_i |\Psi\rangle \,]$$

The difference with empty space is that these are very complicated operators

We only have an **existence proof**.

Then we can construct

$$|B_i\rangle\langle\psi| = Q_{B_i} P_0 Q^\dagger$$

$$|\psi\rangle\langle B_i| = Q P_0 Q^\dagger_{B_i}$$

$$|B_i\rangle\langle B_i| = Q_{B_i} P_0 Q^\dagger_{B_i}$$

$$|\psi\rangle\langle\psi| = Q P_0 Q^\dagger$$

and then construct

Expectation values in state $|\psi\rangle$

$$C_i = |B_i\rangle\langle\psi| + |\psi\rangle\langle B_i| - \langle B_i\rangle\left(|\psi\rangle\langle\psi| - |B_i\rangle\langle B_i|\right)$$

$$\times \frac{1}{\langle B_i^2\rangle - \langle B_i\rangle^2}$$

These operators again have bounded norm and

$$C_i|\psi\rangle = B_i|\psi\rangle$$

so

$$\langle C_{AC}\rangle = \langle C_{AB}\rangle$$

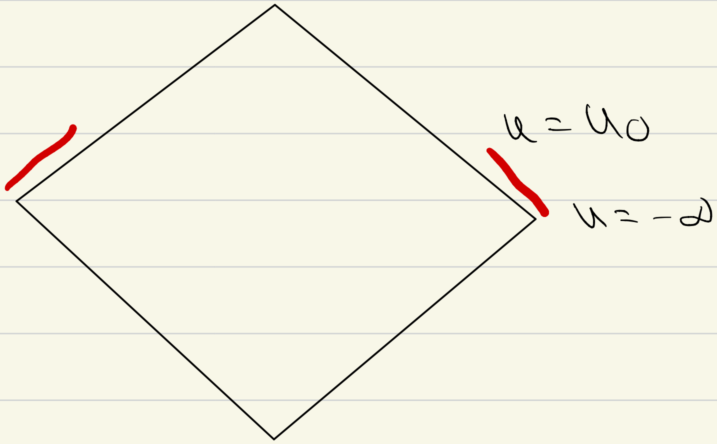
so

$$\langle C_{AC}\rangle^2 + \langle C_{AB}\rangle^2 > 8!$$

Entanglement Entropy at $\mathcal{I}^+$

We now turn to another interesting question

Consider a segment of null infinity



If we think of $\mathcal{I}^+$ as a Cauchy slice, we can ask about the von Neumann entropy of the segment $(-\infty, u_0)$

Formally, this is defined as follows.

Say the system is in some state $|\psi\rangle$

We consider the algebra of operators in $(-\infty, u_0)$: A_{u_0}

We look for an operator

$$\rho \in A_{u_0}$$

so that

$$\mathrm{tr}(\rho O) = \langle\psi|O|\psi\rangle, \ \forall O \in A_{u_0}$$

In the simple case where the Hilbert space factorizes,

$$H = H_{sys} \otimes \tilde{H}_{sys}$$

it is easy to see that this coincides with the "partial trace".

Explanation

Say we have a state $|\psi\rangle \in H$

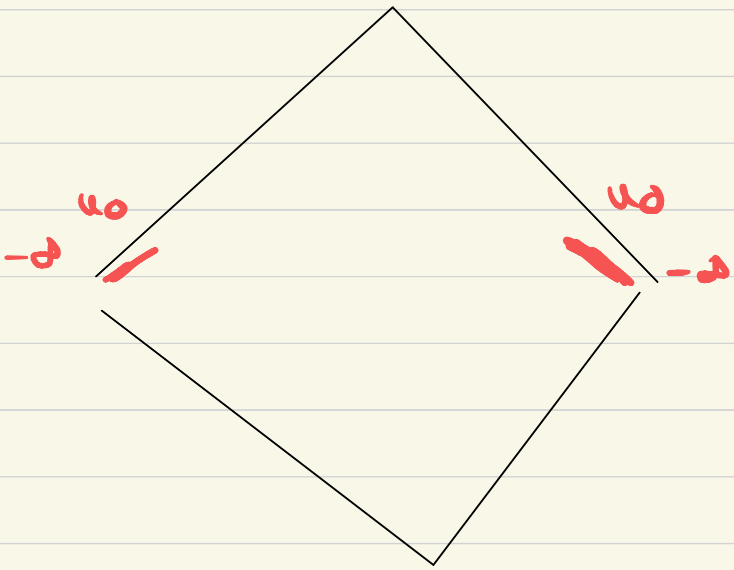
Then $\rho = \mathrm{tr}_{\tilde{H}_{sys}} |\psi\rangle\langle\psi|$ is an operator $\rho : H_{sys} \to H_{sys}$

a) so $\rho \in$ Algebra of operators in H_{sys}

b) Also $\mathrm{tr}(\rho O) = \langle\psi|O|\psi\rangle$, for any $O: H_{sys} \to H_{sys}$ properties (a) and (b) uniquely fix ρ.

Derivation of independence of u_∂

Lets return to the entropy of the segment $(-\partial, u_0)$ of g_t.



We can define an algebra of operators from $(-\infty, u_0)$: A_{u_0}

Let b be an operator from this algebra with the property

$$\mathrm{tr}(b\,O) = \langle O \rangle$$

for any O in A_{u_0}

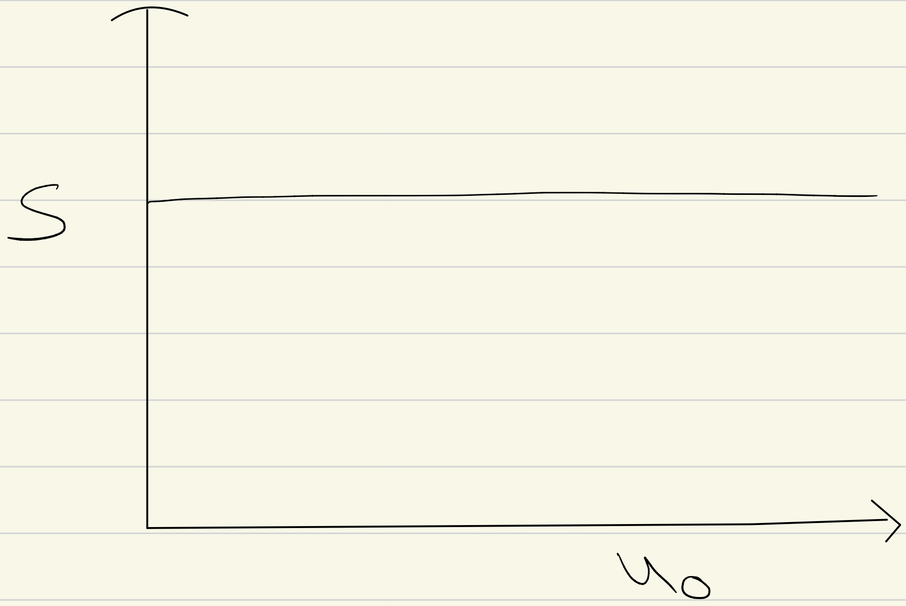
But we showed that any operator at a larger value of u could be approximated by operators in $(-\infty, -\frac{1}{\varepsilon})$

So we can always choose

$$b \in A_{-\alpha}$$

independent of u_0!

This suggests that $S = -\mathrm{tr}(b \ln b)$ is idependent of u_0



Some remarks:

a) In general, we should expect a constant because we have not accounted for massive particles.

So even if the global state is pure, we first need to trace over them and obtain a mixed state for massless particles.

b) This is in contrast to the conventional Page curve

We will turn to the perspective that emerges from the island proposal later.

First we explain some of the physics of this flat page curve.

We would like to address the question:

" why can we meaningfully speak of a Page curve for ordinary objects like coal but not for black holes."

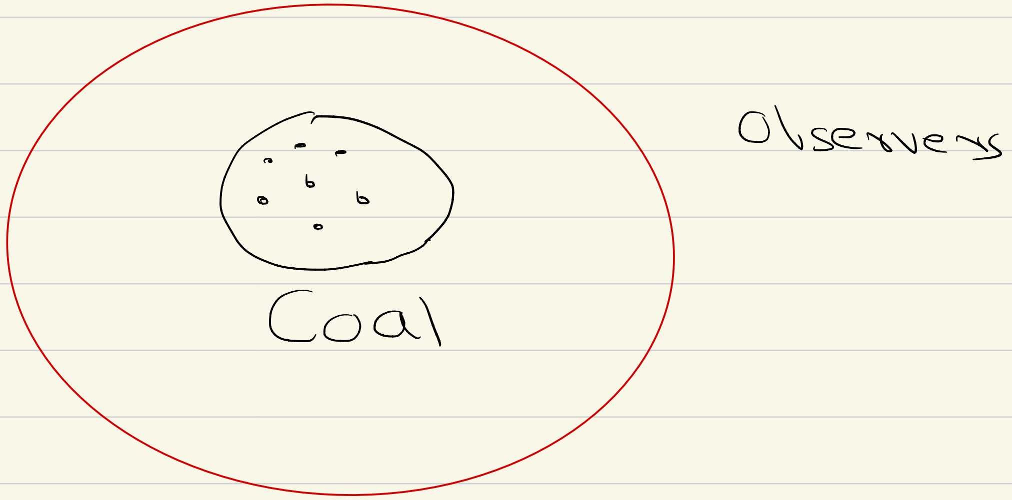
The following discussion is **SUGGESTIVE**

We will make some interesting observations but also describe potential loopholes!

BEGIN SUGGESTIVE PART

Black holes vs coal

The PHoI tells us that even for coal, the information is accessible before the coal burns



But there is a very important distinction!

There is a clear sense in which we expect the F.F. of radiation from coal to follow a Page curve.

The distinction is that to determine the state of the interior of coal using gravitational effects, we need to make measurements to an accuracy controlled by Q.G. effects.

$$O\left(\frac{E}{M_{Pl}}\right), \quad E \sim \text{energy scale of observations}$$

We can consistently take a limit where $M_{Pl} \to \infty$ but the entropy of the coal remains finite.

On the other hand, to determine the microstate through direct measurement requires an accuracy

$$O(e^{-S})$$

[Recall our previous discussion.]

In the limit above, it is clearly easier to directly measure the radiation rather than using Q.G. effects to determine the state of the coal.

On the other hand, such a limit does not exist in any **obvious way** for black holes.

For a black hole

$$S \sim \left(\frac{E}{M_{Pl}} \right)^2$$

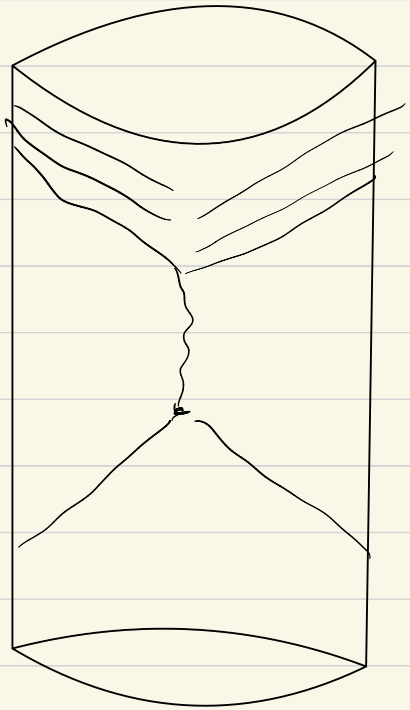
where $E \sim$ typical energy of Hawking quanta.

So there is no obvious <u>hierarchy</u> of difficulty between determining the microstate using these effects and by "collecting" the Hawking radiation.

Emph: doesn't mean such a hierarchy doesn't exist! Just that it <u>may</u> <u>not</u> exist.

Small AdS Black Holes

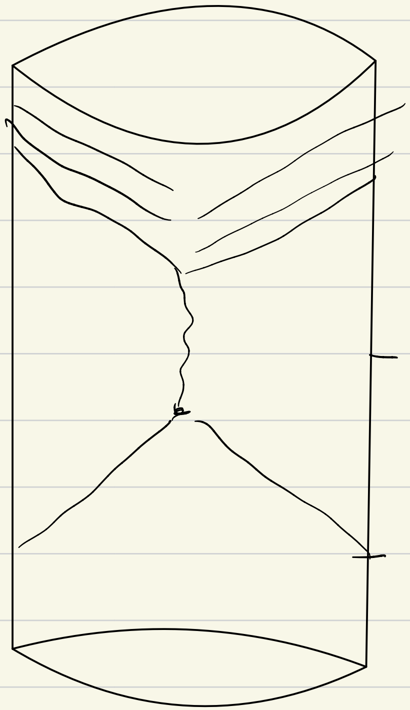
We can see another example with small black holes in AdS



A small black hole is one that is much smaller than the AdS radius. and evaporates.

On the boundary, we can think of the formation and evaporation of a "metastable" state [In $N=4$, we can think of a quark gluon plasma.]

Consider times t_1, t_2, t_3



t_3 → afte b.h. evaporates

t_2 → while b.h. exists

t_1 ← befor b.h. forms

Purely in the CFT, we can ask:

If we want to distinguish the microstate using correlators

$$\langle\psi|\, \underset{\substack{\uparrow \\ \text{CFT ops}}}{O(\tau_1)} \cdots\cdots O(\tau_n)\, |\psi\rangle$$

how does this difficulty change if

τ_i are near t_1, t_2 or t_3.

Simple estimate

1) For τ_i near t_1, it is relatively easy to identify the microstate

2) For τ_i near t_2, it is difficult to identify the microstate.

requires e^{-S} accuracy

[This corresponds to using Q.G. effects to determine the state.]

3) But the entropy of Hawking radiation is **larger** than the B.H.

So, for τ_i near t_3, it may require observations with $e^{-S'}$ accuracy where

$$S' > S$$

is entropy of Hawking rad.

Systems don't "un-thermalize"!

So it may be that waiting for the b.h. to evaporate **increases the difficulty** of reconstruction.

[Possible loophole: small black holes are atypical states, and generic complexity considerations may not apply to them.]

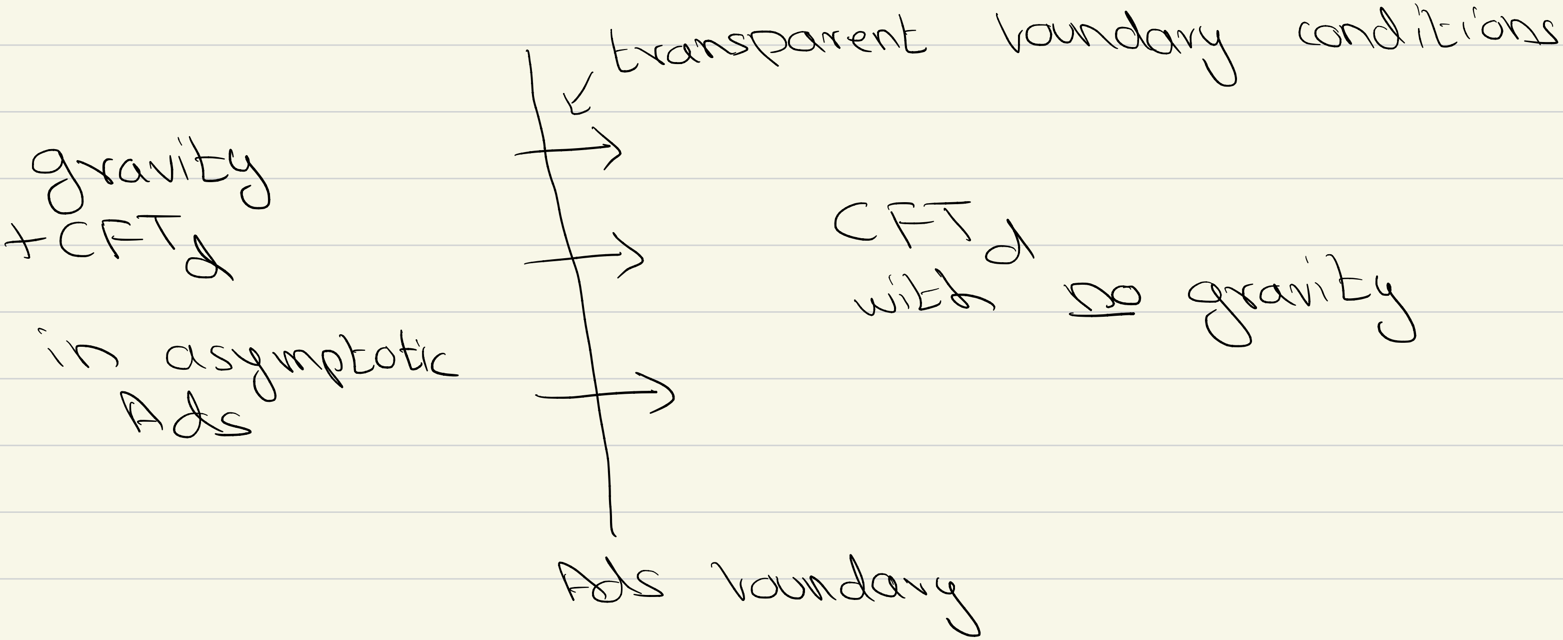
END SUGGESTIVE PART

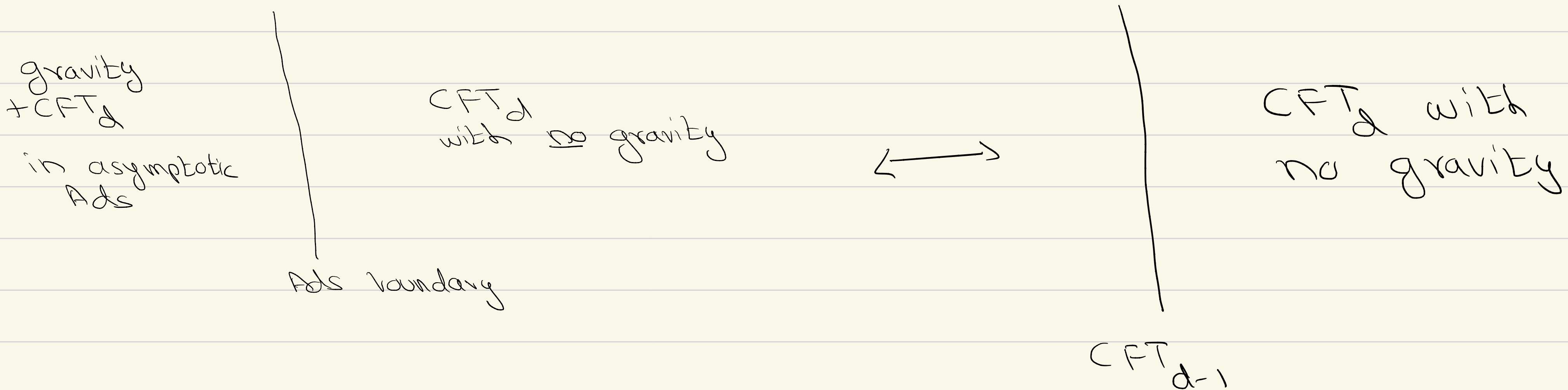
Perspective on the Page curve from islands

There has been significant progress on computing the Page curve in AdS/CFT

These results are **perfectly consistent** with our previous results.

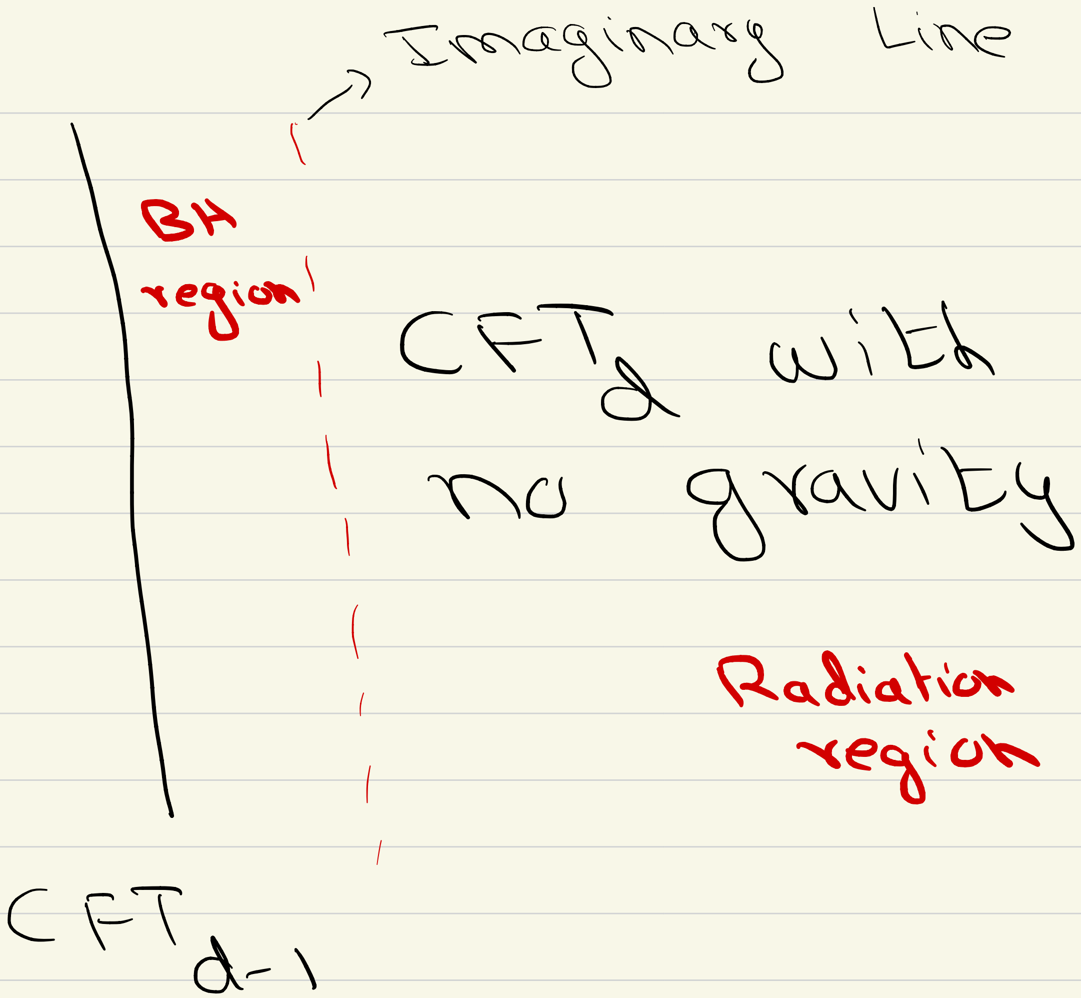
The precise setup is as follows.





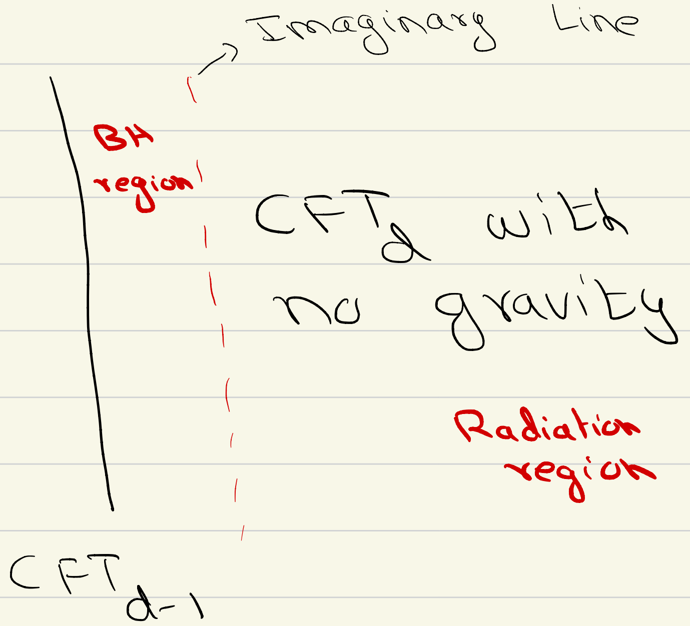
The entire system can be reduced to a CFT$_d$ on a **half-space** ending at a **defect** where a CFT$_{d-1}$ lives.

In this description there is **no gravity**

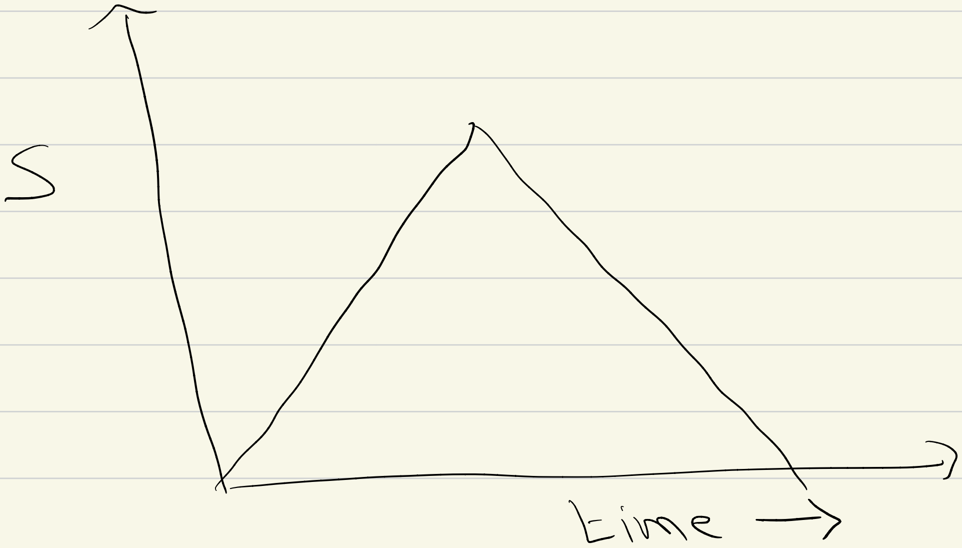


In this nongravitational region, one divides the system into two parts.

We call the region to the left the "black hole". Region to the right "the radiation"



In this nongravitational setup, the Hilbert space factorizes and we see a "Page curve" for the radiation



Comments

1)

The Page curve answers a **nongravitational** question.
But we can use the gravitational dual to answer it.

Similar to AdS/CMT or AdS/Fluid correspondence. We ask a well-defined nongravitational question and use gravity to answer it.

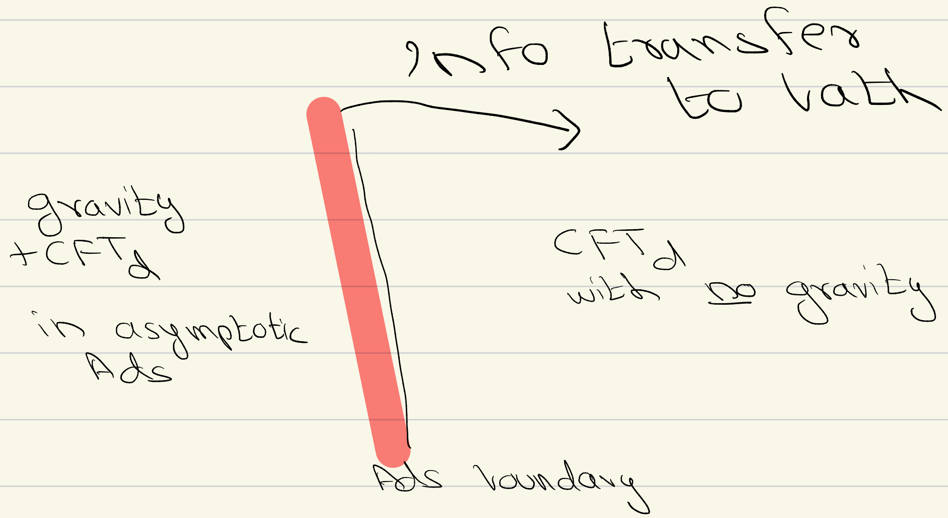
eg. $$\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} \chi(k,\omega).$$

$\uparrow$ two point density correlator

The RHS can be computed using gravity but the "conductivity" makes sense on the bdry

~ the Page curve makes sense on the bdry.

2)



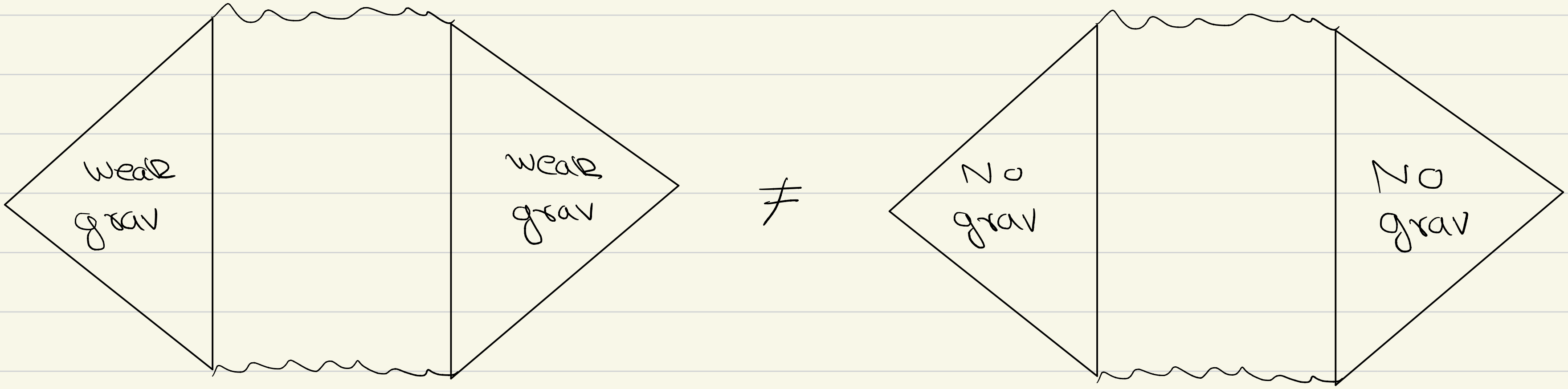
This is consistent with the principle of holography of information.

Info is present near the boundary of AdS. We compute the rate at which it is transferred to the bath.

[Note: we never compute the rate at which information comes out of B.H.]

3) The island computations are technically correct.

But



The words sometimes used

"We go far away from B.H. and then we can ignore gravity"

involve an error we have explored repeatedly

For fine-grained q. info questions, weak grav ≠ no grav