Overview

This set of lectures is geared towards understanding the. modern information paradox.
However, we will start by investigating simple classical properties of black holes. Then consider QFT in curved spacetime and then the information paradox.

Very useful to read some of the G. R. centennial papers collected by the APS.

Rough Overview

Lec 1: Schwarzschild black hole.

Lec 2: other B.H. solutions $\rightarrow$ R.N, kerr, AdS-Schwarzschild, Oppenheimer-Snyder.

Lec 3: Black hole thermodynamics

Lec 4: QFT in curved space, Rindler quantization in flat space

Lec 5: Hawking radiation.

Lec 6: The old Information Paradox & Resolution

Lec 7: The strong Subadditivity Paradox & Resolution

Lec 8: The Modern Info Par & Resolutions

Lec 9: Open Questions.

Deriving the Schwarzschild solution

We will try and find a spherically symmetric solution of the vacuum equations.

Take a metric ansatz:

$$ds^2 = A_0(r_0, t_0)\, dt^2 + B_0(r_0, t_0)\, dr\, dt + C_0(r_0, t_0)\, dr^2$$
$$+ D_0(r_0, t_0)\, d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

and this part is enough to give spherical symmetry.

define a new coordinate

$$r^2 = D(r_0, t_0)$$

Now making a coordinate transformation to this new variable we get

$$ds^2 = A_1(r, t_0)\,dt^2 + B_1(r, t_0)\,dr\,dt + C_1(r, t_0)\,dr^2 + r^2\,d\Omega^2$$

We still have freedom to alter t. We want to make this choice to get rid of the cross term.

$$g_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}\frac{\partial x^\nu}{\partial x^{\bar{\nu}}}\, g_{\mu\nu}$$

$$g_{rt} = \left(\frac{\partial t_0}{\partial r}\right)_t \left(\frac{\partial t_0}{\partial t}\right)_r A_1 + \left(\frac{\partial t_0}{\partial t}\right)_r B_1$$

$$dt = \left(\frac{\partial t}{\partial t_0}\right) dt_0 + \left(\frac{\partial t}{\partial r}\right) dr \Rightarrow \left(\frac{\partial t_0}{\partial r}\right)_t = -\left(\frac{\partial t}{\partial r}\right) \Big/ (\partial t / \partial t_0)$$

So we need to solve

$$A_1 \left(\frac{\partial t}{\partial r}\right) / \left(\frac{\partial t}{\partial t_0}\right) = B_1$$

given some initial condition $t(r)$ at $t_0 = 0$, we can solve this systematically.

So we have reduced our ansatz to the form

$$ds^2 = A(r,t)\, dt^2 + B(r,t)\, dr^2 + r^2 d\Omega^2$$

Now we need to input dynamics. So we use the vacuum Einstein equations.

$$R_{\mu\nu} = 0$$

[Derive this in the tutorial starting with ordinary eqns.]

We compute the Ricci tensor. We find [Check in tutorial using computer programs]

$$R_{tt} = R_{rr}\frac{A}{B} - \frac{B\frac{\partial A}{\partial r} + A\frac{\partial B}{\partial r}}{rB^2}$$

$$R_{tr} = \frac{\frac{\partial B}{\partial t}}{rB} \; ; \quad R_{\theta\theta} = 1 - \frac{1}{B} - \frac{r\frac{\partial A}{\partial r}}{2AB} + \frac{r\frac{\partial B}{\partial r}}{2B^2}$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

Note immediately that $R_{tt} = R_{rr} = 0$

$$\Rightarrow \quad B\frac{\partial A}{\partial r} + A\frac{\partial B}{\partial r} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(AB) = 0$$

Also $R_{tr} = 0 \quad \Rightarrow \quad \dfrac{\partial B}{\partial t} = 0$

So the equation remaining to solve is $R_{\theta\theta} = 0$

This can be solved through

$$A = -\left(1 - \frac{2m}{r}\right) \; ; \quad B = \frac{1}{1 - \frac{2m}{r}}$$

[Tutorial: check this solution. Also ask why we can't take $AB = f(t)$, rather than $AB = -1$; ans: f can be eliminated through a suitable redef. of t.

To see the physical significance of m, consider the asymptotic form of the geodesic equation,

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\,\dot{x}^\nu \dot{x}^\rho = 0$$

In the low velocity limit, the only relevant term comes from $\nu = \rho = t$ and with $\mu = r$, we find

$$\frac{d^2 r}{d\tau^2} + \frac{m}{r^2} = 0$$

$$\Rightarrow \quad \frac{d^2 r}{d\tau^2} = -\frac{m}{r^2} \quad \Rightarrow \quad m = GM.$$

To summarize, the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\Omega^2$$

with $m = GM$. [Tutorial: restore factors of c. Compute m for the sun. Ans: about 3km]

We have proved a version of Birkhoff's theorem here

The Schwarzschild solution is the unique spherically symmetric vacuum solution. Spherical symmetry automatically leads to a static solution.

Red-shifts

The first significant physical effect we see is that of the red shift. Consider observers at radial positions r_1, r_2.

the person at r_1 is sending light rays to the person at r_2.

The rays travel along

$$dr^2\left(1-\frac{2m}{r}\right)^{-1} - \left(1-\frac{2m}{r}\right)dt^2 = 0$$

but crucially, the coordinate time δt measured by both is equal.

but the proper time is different. For observer 1, the proper time is

$$\delta\tau_1 = \sqrt{\left(1-\frac{2m}{r_1}\right)}\,\delta t$$

and for 2, it is

$$\delta\tau_2 = \sqrt{\left(1-\frac{2m}{r_2}\right)}\,\delta t$$

so

$$\frac{\delta\tau_1}{\delta\tau_2} = \sqrt{\frac{\left(1-\frac{2m}{r_1}\right)}{1-\frac{2m}{r_2}}} \quad \text{or} \quad \frac{\omega_1}{\omega_2} = \sqrt{\frac{\left(1-\frac{2m}{r_2}\right)}{\left(1-\frac{2m}{r_1}\right)}}$$

Tutorial: Consider two observers. One lives at sea level the other lives on a 3 km high mountain. When the sea-level person ages 70 yrs, how much does the mountain person age?

Now we turn to investigate the region $r=2m$. Notice that the metric is becoming singular here. But this is only a coordinate singularity [See Kruskal's paper in G.R. centennial collection.]

The first step is to define the so-called tortoise coordinate so that the metric looks like

$$ds^2 = f(r_*)\left[-dt^2 + dr_*^2\right] + g(r_*)\,d\Omega^2$$

clearly we need,

$$\left(1-\frac{2m}{r}\right) dr_*^2 = \frac{dr^2}{1-\frac{2m}{r}} \;; \qquad dr_* = \frac{dr}{1-\frac{2m}{r}}$$

which leads to

$$r_* = r + 2m \ln \left| \frac{r-2m}{2m} \right|$$

as

$$r \to 2m, \quad r_* \to -\infty.$$

Next we define

$$U = -e^{\alpha(r_* - t)} \; ; \quad V = e^{\alpha(r_* + t)}$$ will fix α below.

then

$$dU\, dV = -\alpha^2 (dr_* - dt)(dr_* + dt)\, e^{2\alpha r_*}$$

Now near the horizon

$$e^{2\alpha r_*} = e^{2\alpha r_h} \left(\frac{r-2m}{2m} \right)^{4m\alpha}$$

So if we set $4m\alpha = 1$, we see that the near horizon metric becomes proportional to $dU\, dV$.

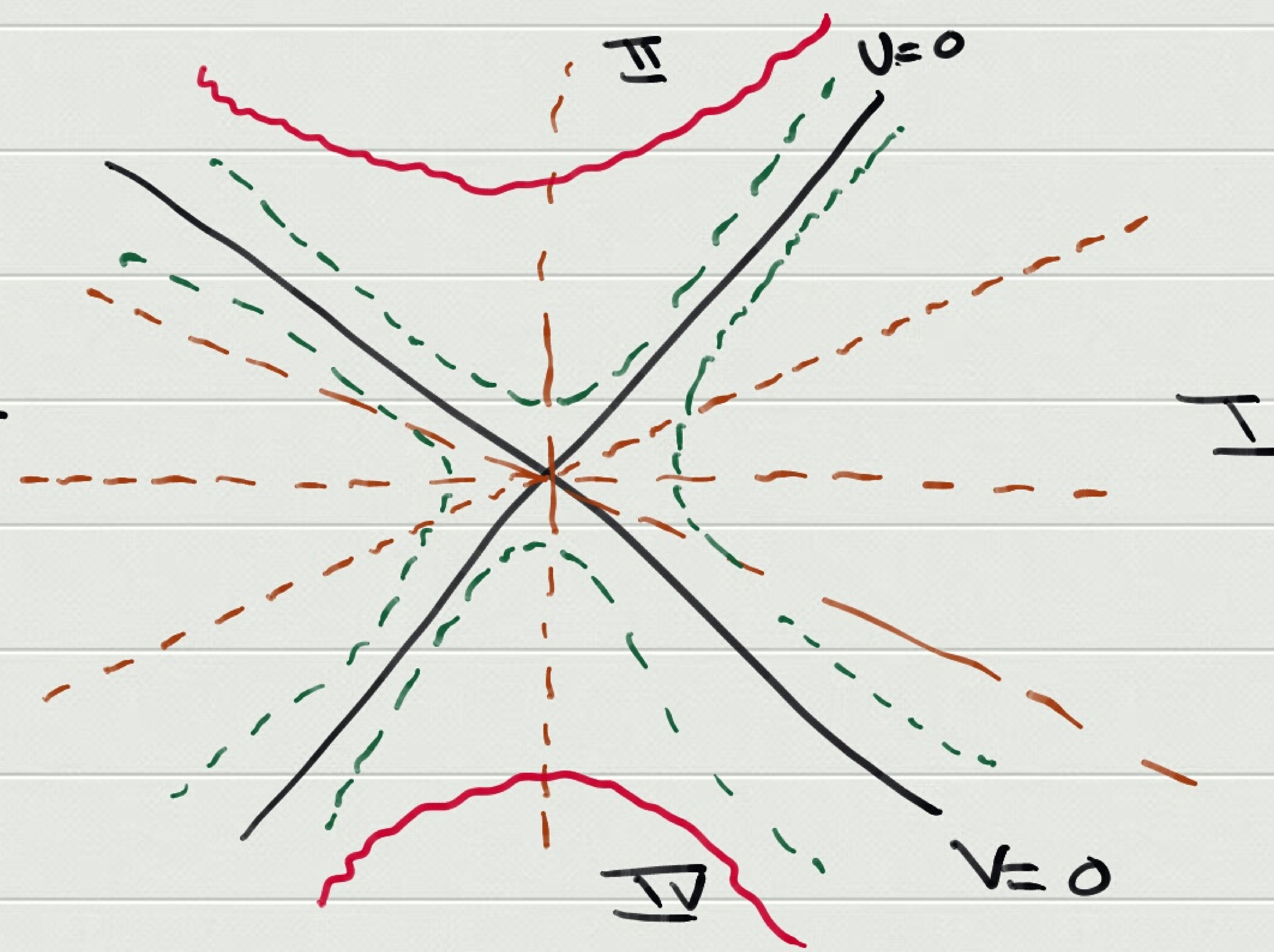
More precisely

$$ds^2 = -dU\,dV\left(\frac{32M^3 e^{-r/2m}}{r}\right) + r^2 d\Omega^2$$

which is manifestly non-singular as $r \to r_h$.
The horizon, in these coordinates is $U = 0$. We can smoothly extend the metric into the region $U > 0$.
There is, however, a <u>true</u> <u>singularity</u> at $r = 0$.

[we will check this later.]

This new coordinate system gives us a picture of the kruskal extension of the Schwarzschild black hole.



In regions II $(U>0, V>0)$, we can introduce another patch through $U = e^{\alpha(r_*-t)}$; $V = e^{\alpha(r_*+t)}$; in region III $(U>0, V<0)$, $U = e^{\alpha(r_*-t)}$; $-V = e^{\alpha(r_*+t)}$

In region $\overline{IV}$, $(U<0,\ V<0)$

$$-V = e^{\alpha(r_* - t)} \quad ; \quad -V = e^{\alpha(r_* + t)}$$

Note that constant $t \Rightarrow$ constant $\dfrac{V}{U}$

$r =$ constant $\Rightarrow UV =$ const.

So $r=$const. are hyperbolids in the U, V plane whereas $t =$ const. are straight lines passing through the origin.

The point where $r=0$ is where $r_*=0$ and $UV = 1$.

This is the singularity.

To check that $r=2m$ is non-singular and that $r=0$ is singular, we can consider curvature invariants

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6}$$

(Recall $R=0$ throughout spacetime)

In particular at $r=2m$, we see that

$$(R_{\alpha\beta\gamma\delta})^2 \rightarrow \frac{1}{m^4}$$

So large black holes have smooth horizons!

$r=0$ is a curvature singularity.

The last topic to cover is the Penrose diagram. This helps visualize the causal structure by compactifying the spacetime. define

$$x = \tan^{-1} U; \qquad \psi = \tan^{-1} V$$

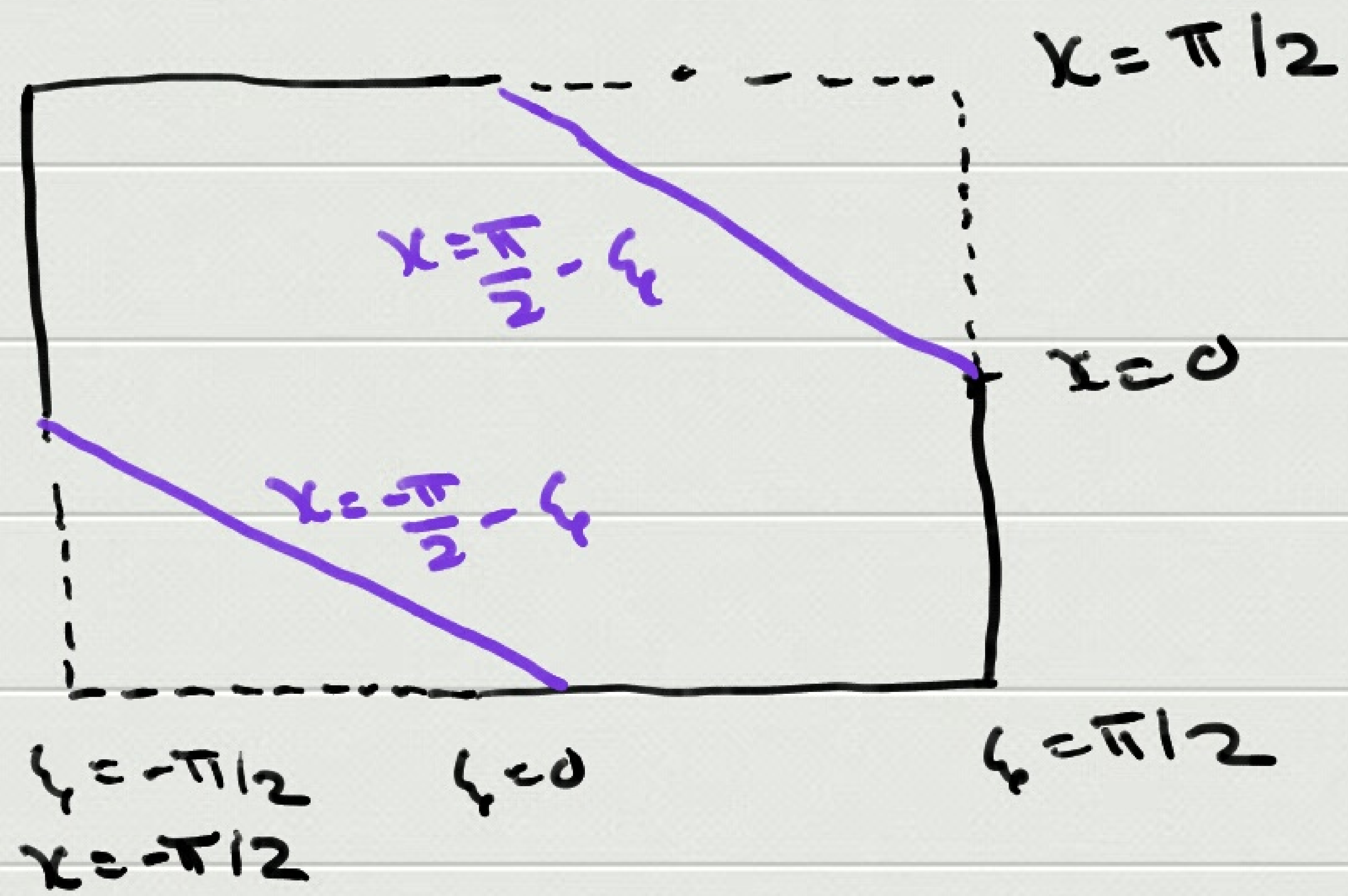
then $\quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; \quad \psi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

except for the cutoff at $\quad UV = 1 \quad$ or

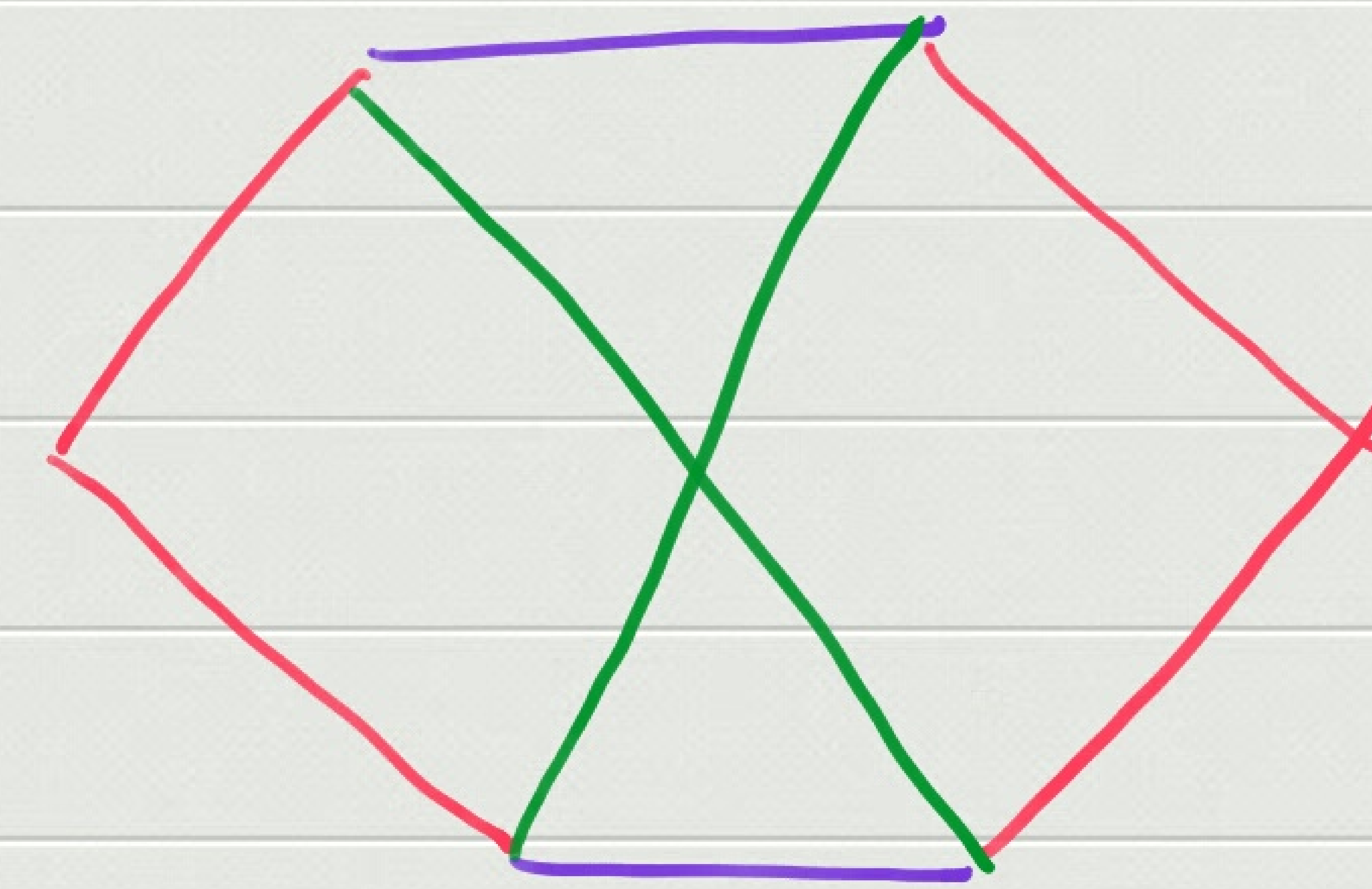
$\tan x \tan \psi = 1 \quad$ or $\quad \psi = \frac{\pi}{2} - x$

or $\psi = -\left(\frac{\pi}{2} + x\right)$.

leads to



Rotate 45° $\longrightarrow$
to make nullrays at 45°



The green lines are <u>horizons</u>. They divide the spacetime into two disconnected parts

[experience of the infaller in his frame and according to an external observer.]