14 April 2021

Lecture 25
More on State dependence

Last time we introduced the notion of state-dependent operators.

The idea was that the mirror operators that describe "right movers" behind the horizon might depend on the microstate.

More precisely in the little Hilbert space

$$H_\psi = \text{Span}\{A_i|\psi\rangle\}$$

$\tilde{a}_\omega$ acts linearly and like an ordinary operator

But we might need a different operator about a different microstate.

Resolving paradoxes.

If we follow this idea, we see that it resolves all paradoxes associated with large black holes.

Negative occupancy paradox

For instance, we clearly have.

$$\langle \psi | \tilde{a}_\omega \tilde{a}_\omega^\dagger | \psi \rangle = \frac{1}{1 - e^{-\beta\omega}}$$

But we cannot use

$$\mathrm{tr}\left(e^{-\beta H} \tilde{a}_\omega \tilde{a}_\omega^\dagger\right) = \sum_E \langle E | e^{-\beta H} \tilde{a}_\omega \tilde{a}_\omega^\dagger | E \rangle$$

$$\neq \sum_E \langle E | \tilde{a}_\omega^\dagger e^{-\beta H} \tilde{a}_\omega | E \rangle$$

Because the operator $\tilde{a}_\omega$ <u>depends</u> on the state $|E\rangle$.

Note that the trace is not directly observable.

So this dependence is not directly observable as emphasized earlier.

The same resolution works for the paradox with the infalling number operator

Recall

$$N_\alpha \propto (a^\dagger_\omega - e^{-\beta\omega/2}\tilde{a}_\omega)(a_\omega - e^{-\beta\omega/2}\tilde{a}^\dagger_\omega)$$
$$+ (\tilde{a}^\dagger_\omega - e^{-\beta\omega/2} a_\omega)(\tilde{a}_\omega - e^{-\beta\omega/2} a^\dagger_\omega)$$

Clearly

$$N_a |\Psi\rangle = 0$$

for an equilibrium state.

But it is also true that

$$\langle n | N_a | n \rangle \neq 0$$

where $|n\rangle$ is a Schwarzschild number eigenstate.

Since N_a is state dependent

$$\sum_E \langle E | N_a | E \rangle \neq \sum \langle n | N_a | n \rangle .$$

Finally let us consider the paradox with the eternal black hole.

The paradox was that we were unable to find operators $\tilde{O}_\omega$ that had the right correlators in states

$$e^{i(H_L+H_R)T}|\Psi_{tfd}\rangle.$$

We can simply use the construction of the mirror operators.

But there is another approach that yields insight into the origins of state dependence.

Recall that if we use the modes of the left CFT $a_{\omega,L}$ then we do have

$$\langle \Psi_{tfd} | \, a_{\omega,L} \, a_{\omega} \, | \Psi_{tfd} \rangle = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}}$$

The problem is

$$\langle \Psi_{tfd} | \, e^{i(H_L + H_R)T/2} \, a_{\omega,L} \, a_{\omega} \, e^{-i(H_L + H_R)T/2} \, | \Psi_{tfd} \rangle$$

$$= e^{-i\omega t} \, \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega/2}}$$

Let

P_T be the projector on the little Hilbert space constructed about $e^{-i(H_L+H_R)T/2}|\psi_{tfd}\rangle$

then

$$\tilde{a}_\omega = \sqrt{\frac{C}{\pi\beta^2}} \int_{-\Theta_{cut}}^{\Theta_{cut}} d\tau \, a^1_\omega \, e^{i\omega\tau} P_\tau$$

has the right correlators in the thermofield double state and nearly time-shifted states.

The measure is designed using

$$|\langle\psi_T|\psi\rangle|^2 = e^{-T^2 C/\beta^2}$$

where C is the <u>specific heat</u>

The measure is defined so that the projector acts like a delta-function

$$\tilde{a}_\omega A_i |\psi_T\rangle = a_{\omega,L} e^{i\omega T} A_i |\psi_T\rangle$$

The additional factor of $e^{i\omega T}$ is exactly what we need

<u>Why is this state dependent?</u>

Note the cutoff, Θ_{cut}

$$\tilde{a}_\omega = \sqrt{\frac{c}{\pi\beta^2}} \int_{-\Theta_{cut}}^{\Theta_{cut}} d\tau \, a_\omega^L e^{i\omega\tau} P_\tau$$

This is necessary because if we try and take $\Theta_{cut} \to \infty$, then we run into trouble

First note

$$\langle \psi_{tfd} | P_\tau | \psi_{tfd} \rangle \sim O(e^{-S})$$

Since we are integrating over an infinite range of τ, this "fat tail" means that even

$$\tilde{a}_\omega | \psi_{tfd} \rangle$$

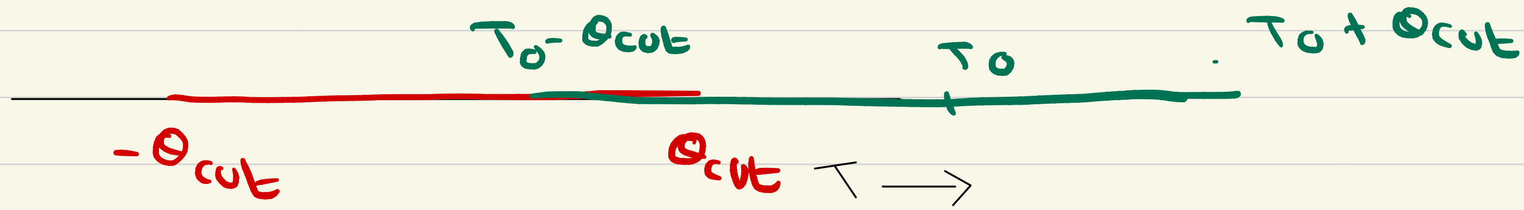
receives a contribution from large τ and so even

$$\langle \psi_{tfd} | a_\omega \tilde{a}_\omega | \psi_{tfd} \rangle$$

ceases to have the correct value if

$$\Theta_{cut} \to \infty$$

The picture is as follows. We are considering a 1-parameter family of states



We can use a single operator for an exponentially long interval. [marked in red]

But eventually, we need to switch to a new operator that again works for a long interval [marked in green.]

We see that the paradoxes for the eternal black hole do not arise since, to obtain the paradox, we assumed that $\tilde{a}_\omega$ was the same operator on $|\Psi_{tfd}\rangle$ and

$$e^{i(H_L+H_R)T/2}|\Psi_{tfd}\rangle$$

for arbitrarily long T.

This is not true if $\tilde{a}_\omega$ is state dependent.

Consistency of state-dependent maps

Although state dependence is very effective in resolving puzzles, it must be carefully checked for consistenc.

[Explain significance]

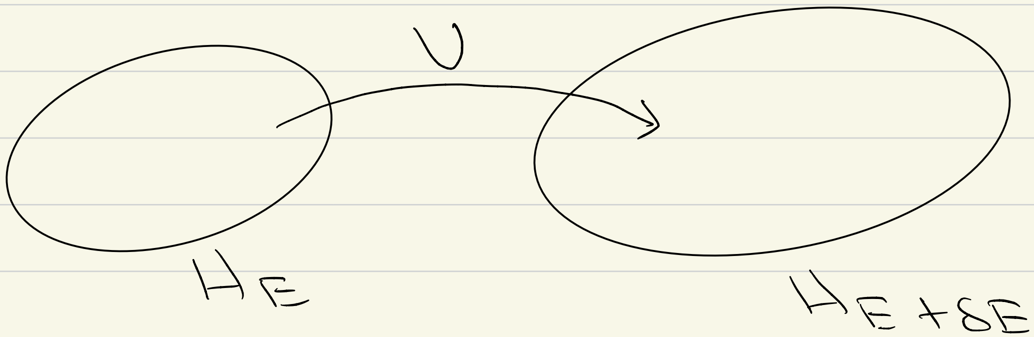
One such puzzle was pointed out in arxiv: 1506.01337

We will describe a generalization and reformulation from 1604.03095.

The main physical point is that if A is a state-independent observable, then one can derive some constraints on how much it changes under a low energy excitation

The result is as follows. Consider a typical state at energy E and a unitary operator, U that increases the energy by δE

[Unitaries must increase the energy of typical states since they do not annihilate any state so they have to map the space at energy E to the slightly larger space at energy $E+\delta E$]



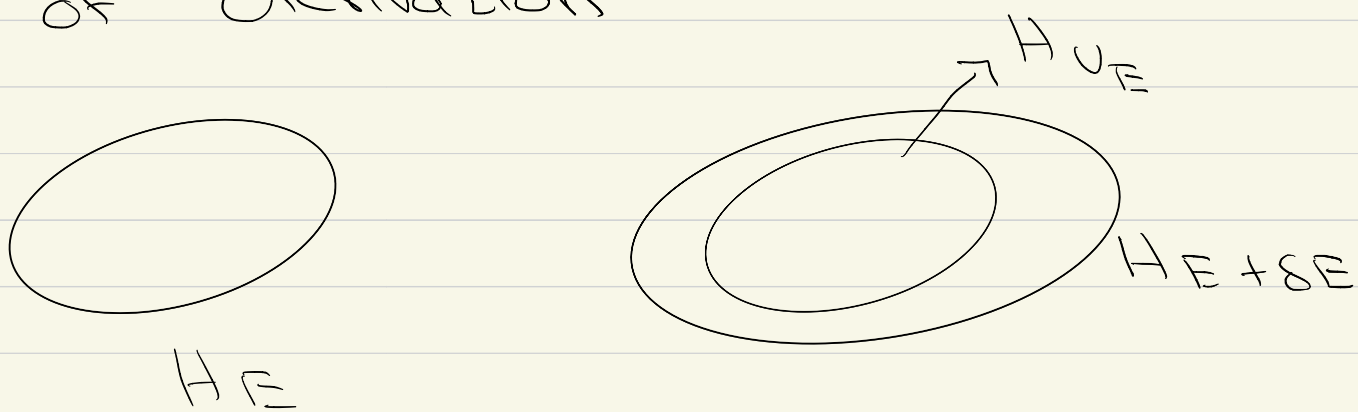
Then the result is that if A is some Hermitian observable then

$$\delta A \equiv \left| \langle \psi | U^{\dagger} A U | \psi \rangle - \langle \psi | A | \psi \rangle \right| \leq 2\sqrt{\beta \delta E}\,\sigma$$

when $\beta \delta E << 1$ and

$$\sigma^2 = \langle \psi | A^{\dagger} A | \psi \rangle - \langle \psi | A | \psi \rangle^2$$

Sketch of derivation



Let H_{v_E} be the image of H_E in $H_{E+\delta E}$

H_{v_E} has dimension e^S

$H_{E+\delta E}$ has dimension $e^{S+\beta\delta E}$

So a vector in $H_{E+\delta E}$ can be written as

from H_{v_E} orth to H_E

$$|\psi'\rangle = \left(1-\frac{\beta\delta E}{2}\right)|\psi_E\rangle + \sqrt{\beta\delta E}\,|\psi_0\rangle$$

Since the temperature associated with $E+\delta E$ is also $\frac{1}{\beta}$

$$\delta\beta = \frac{\partial\beta}{\partial E}\,\delta E = -\beta^2 \frac{\delta E}{C_v} = O\left(\frac{1}{S}\right)$$

specific heat extensive.

so

$$\langle\psi'|A|\psi'\rangle_{\text{typical}} = \langle\psi|A|\psi\rangle_{\text{typical}} \quad \text{for typical states}$$

Also

$$\langle\psi_E|A|\psi_E\rangle = \langle\psi|U^{+} A U|\psi\rangle$$

so from decomposition of $|\psi'\rangle$

$$\left|\langle\psi|A|\psi\rangle - \langle\psi|U^{+} A U|\psi\rangle\right| = O(\sqrt{\beta\delta E})$$

Some more work yields the precise bound.

The idea of the puzzle is as follows

Consider the Schwarzschild number operator

$$N_\omega = a^\dagger_\omega a_\omega$$

This operator has almost 0 energy but not quite because a_ω is a slightly smeared mode.

Now consider

$$U = e^{i\theta N_\omega}$$

This operator has very low energy because N_ω almost commutes with H.

But consider its effect on the correlator

$$\langle\psi|\tilde{a}_\omega a_\omega|\psi\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

Since, $U^\dagger a_\omega U = e^{-i\theta} a_\omega$; $U^\dagger \tilde{a}_\omega U = \tilde{a}_\omega$

we have

$$\langle\psi| U^\dagger \tilde{a}_\omega a_\omega U |\psi\rangle = \frac{e^{-i\theta} e^{-\beta\omega/2}}{1-e^{-\beta\omega}}$$

So the correlator is altered by a large amount and not a small amount.

There is a partial resolution to this problem outlined in 1604.03095

So far we have described simple operators as polynomials in the modes a_ω, $\tilde{a}_\omega$ etc.

But physically how does one perturb a state by U and make an observation.

1) We deform the boundary Hamiltonian by a <u>local</u> simple operator $Q(t)$

$$H(t) = H + J(t)\,Q(t)$$

Note the condition on the operator being "local" in time.

This modifies the state at time t on the boundary as

$$|\psi(t)\rangle = T\left\{e^{-\int_{-\infty}^{t} J(\tau)Q(\tau)d\tau}\right\}|\psi\rangle. \text{ I}$$

It is important that $Q(t)$ be a simple operator.

Otherwise if $Q(t)$ is arbitrary any unitary can be written in the form I

eg we can write.

$$\phi(t)\,\phi(t-T_1)\,\phi(t+T_2)$$

$$= \phi(t)e^{-iHT_1}\phi(t)e^{-iHT_1}e^{-iHT_2}\phi(t)e^{iHT_1}$$

so it is an operator at time t.
But this is not a simple operator

2) We have considered arbitrary correlators of "simple" operators.

But not all are easily observable. For instance

For instance consider the points x_1 and x_2 marked above.

It appears that

$$\langle \psi | \phi(x_1) \phi(x_2) | \psi \rangle$$

is a good observable.

But it cannot be observed by a simple bulk observer because even if observers at x_1 and x_2 signal each other at the speed of light, those light signals cannot meet before the singularity.

"A set of points is in a causal patch if future directed null/timelike geodesics from the points can meet at a common point"

In figure x_1, x_3 are in a causal patch.

So we can further restrict wlk observables to

$$\langle \psi | \phi(x_1) \ldots \phi(x_n) | \psi \rangle$$

where x_i are in a single causal patch.

[Note: Entire discussion is in the context of QFT in curved spacetime, where we neglect "holography of information" and other aspects of subtlety with locality in gravity.

Appropriate perspective for <u>leading</u> order field correlators.]

Then it is a very remarkable property of AdS correlators that for

$$U = e^{i \int_{-\infty}^{t_c} J(t) Q(t)}$$

and

$$A = \phi(x_1) \ldots \phi(x_n)$$

where x_i are in the causal patch that ends at t_c on the boundary

We do have

$$\langle \psi | U^{\dagger} A U | \psi \rangle - \langle \psi | A | \psi \rangle \leq \sqrt{\beta S E}\, \sigma$$

I don't know of any simple way to motivate this result but it follows from an analysis of position-space AdS correlators.

What this means is that when we further restrict simple observables by using bulk locality to ask which observables can be measured and which perturbations can be generated dynamically, the "anomalous sensitivity of black holes to low energy excitations" goes away!

But this is not a complete resolution to the paradox.

Consider the original paradox framed in terms of

$$e^{i\theta N_\omega}$$

where

$$N_\omega = a_\omega^\dagger a_\omega$$

in position space, we would write

$$N_\omega = \frac{1}{G_\omega} \int O(t) e^{-i\omega t} \int O(t') e^{i\omega t'}$$

c-number

To act with this operator at $t=0$ requires operators "from the future"

So the resolution does not apply to deformations of this kind. We declare that these are unobservable in simple experiments.