14 Jan 2021 Lecture 2

We concluded Lecture 1 with an idea for how to extract modes across a null surface
[Brief Recap]

To extract these modes precisely, we need the following

1) We don't want the mode to have support at large $|U|$

2) we also need to cut it off near $U=0$

So we introduce a smearing function $T(U)$ that dies off smoothly near $U \to 0$, for $U < U_l$ and for "larger" U, $U > U_h$

The precise details of $T(U)$ will never be important

Only some of its general properties.

1) we need $U_\ell << U_h$ and $U_h << \ell_{\text{curvature}}$

2) Precise normalization is

$$\int T(U)^2 \frac{dU}{U} = 2\pi$$

$T(U)$ is flat for a large range of $\log U$

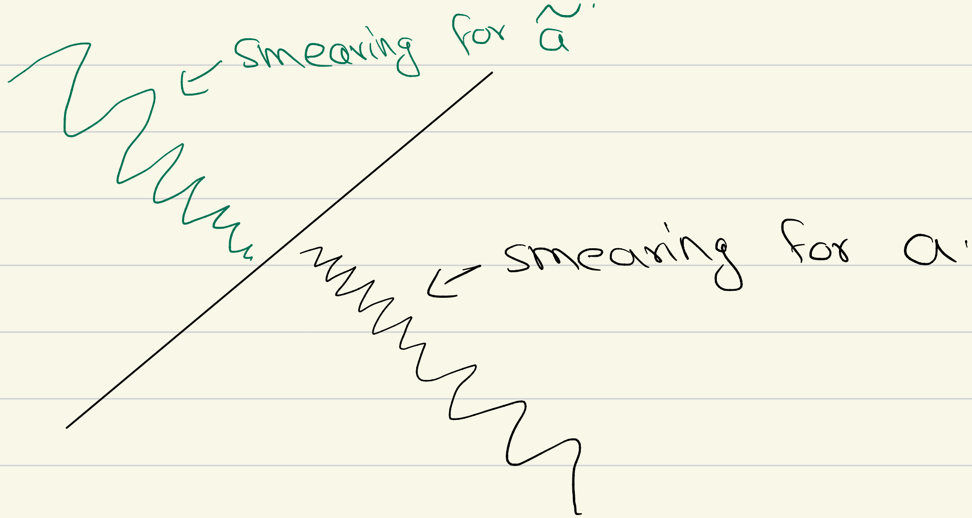
We also need to smear a little in the transverse direction. We smear in a small volume Vol.

Then define

$$a = \int \partial_U \phi(U, V=0, y^\alpha)\,(-U)^{-i\omega_0}\,T(-U)\,dU\,\frac{d^{d-1}y^\alpha}{\sqrt{\pi\,\omega_d\,\mathrm{Vol}}}$$

$$\tilde{a} = \int \partial_U \phi(U, V\underline{=-\varepsilon}, y^\alpha)\,U^{i\omega_0}\,T(U)\,dU\,\frac{d^{d-1}y^\alpha}{\sqrt{\pi\,\omega_d\,\mathrm{Vol}}}$$

Note separation in V



$a^\dagger$ and $\tilde{a}^\dagger$ are defined by conjugation.

These modes depend on

1) A choice of origin [defined locally]
2) A choice of frequency ω_0

The correlators of these modes depend only on the short distance correlators of the field.

For instance

$$\langle a \tilde{a} \rangle = \underline{\underline{\frac{1}{\pi \mathrm{Vol}\, \omega_0}}} \int dU_1\, dU_2 \, \langle \partial_{U_1} \phi(U_1, V=0, y)\, \partial_{U_2} \phi(U_2, V=-\epsilon, y_2) \rangle (-U_1)^{-i\omega_0} U_2^{i\omega_0} \, T(-U_1) T(U_2)\, d^{d-1} y_1\, d^{d-1} y_2$$

From $\sqrt{\ }$ factors

$$= \frac{-1}{4\pi^2 \omega_0} \int \frac{1}{(U_1 - U_2)^2} \left(\frac{U_2}{-U_1} \right)^{i\omega_0} T(-U_1) T(U_2)\, dU_1\, dU_2$$

One way to do this integral is to note the identity

$$\frac{1}{(v_1-v_2)^2} = \frac{1}{(-v_1)\,v_2}\int_{-\infty}^{\infty} \frac{w\, e^{-\pi w}}{1-e^{-2\pi w}}\left(\frac{v_2}{-v_1}\right)^{-iw} dw$$

when $v_1 < 0$ and $v_2 > 0$

To see this note when $|v_1| > |v_2|$ we pick up the poles at $w = in$, leading to

$$\frac{1}{|v_1|\,v_2}\sum_{n=1} -n\cdot(-1)^n\left(\frac{v_2}{|v_1|}\right)^{in\cdot}$$

no pole at $n=0$

factor of 2π from residue thm cancels with factor from $1-e^{-2\pi w} \to 2\pi(w-in)$
factor of i combines for $-$ sign.

$$= \frac{1}{(v_1-v_2)^2}$$

[When $|v_2| > |v_1|$ close contour to pick up poles at $w = -in$]

Returning to the integral we started with.

$$\frac{-1}{4\pi^2\omega_0}\int \frac{1}{(U_1-U_2)^2}\left(\frac{U_2}{-U_1}\right)^{i\omega_0} T(-U_1)T(U_2)\, dU_1\, dU_2$$

$$= \frac{1}{4\pi^2\omega_0}\int \frac{dU_1}{U_1}\frac{dU_2}{U_2}\frac{\omega\, e^{-\pi\omega}}{1-e^{-2\pi\omega}}\left(\frac{U_2}{-U_1}\right)^{-i\omega}\left(\frac{U_2}{-U_1}\right)^{i\omega_0} T(-U_1)T(U_2)$$

Using the Fourier inversion theorem, this becomes

$$\frac{1}{\omega_0}\int \omega\frac{e^{-\pi\omega}}{1-e^{-2\pi\omega}}\left|s(\omega-\omega_0)\right|^2 d\omega$$

where $s(r) = \frac{1}{2\pi}\int_0^\infty \frac{dU}{U}\, T(U)\, U^{-ir}$.

In the limit under consideration $s(r)$ is strongly peaked around $r=0$

So we find

$$\langle a \tilde{a} \rangle = \frac{e^{-\pi \omega_0}}{1 - e^{-2\pi \omega_0}}$$

Similar calculations yield

$$\langle a a^{+} \rangle = \frac{1}{1 - e^{-2\pi \omega_0}}$$

$$\langle \tilde{a} \tilde{a}^{+} \rangle = \frac{1}{1 - e^{-2\pi \omega_0}}$$

Assignment questions

Also

$$[a, a^{+}] = 1 = [\tilde{a}, \tilde{a}^{+}]$$

$$[a, \tilde{a}] = 0$$

follows from field commutators.

A few important physical points

a) There is a dependence on the parameter ω_0 which we are suppressing

b) Dependence on a point about which the modes are defined

c) There is no δ-fn in these correlators. Distinguish them from

$$\langle a_\omega \tilde{a}_{\omega'} \rangle = \frac{e^{-\pi\omega'}}{1-e^{-2\pi\omega'}}\,\delta(\omega-\omega')$$

Nevertheless if we take a distinct frequencies ω_0 and define modes b then

$$\langle a b^\dagger \rangle = 0 .$$

[width of the modes is controlled by $T(\omega)$]

We need one last technical refinement.

The modes so far were defined by smearing in a null coordinate and in a volume Vol.

Often it is easier to deal with spacetimes with spherical symmetry

So we assume that the metric can be written about some sphere as

$$-dU\,dV + r_0^2\, d\Omega_{d-1}^2 + \ldots$$

← terms relevant away from $U=0, V=0$

Then we can define

$$a = \frac{r_0^{d-1}}{\sqrt{\pi\omega_0}} \int \partial_U \phi(U, V=0, \Omega)(-U)^{-i\omega_0} T(-U)\, dU\, Y_l^*(\Omega) d\Omega$$

note opposite convention

$$\tilde{a} = \frac{r_0^{d-1}}{\sqrt{\pi\omega_0}} \int \partial_U \phi(U, V=-\varepsilon, \Omega)\, U^{i\omega_0} T(U)\, Y_l(\Omega)\, d\Omega$$

These modes depend on a spherical harmonic and not a point.
Here "l" is used as shorthand for all required quantum numbers.

These modes satisfy the same relations

$$\langle a \tilde{a} \rangle = \frac{e^{-\pi\omega_0}}{1 - e^{-2\pi\omega_0}}$$

$$\langle a a^{+} \rangle = \frac{1}{1 - e^{-2\pi\omega_0}}$$

$$\langle \tilde{a} \tilde{a}^{+} \rangle = \frac{1}{1 - e^{-2\pi\omega_0}}$$

$$[a, a^{+}] = 1 = [\tilde{a}, \tilde{a}^{+}]$$

So far we have discussed correlators. But we can make a stronger statement about the relationship of these modes.

We have been writing $\langle\ \rangle$ for these correlators but we really mean that we are considering correlators in some state $|\psi\rangle$.

In the full theory, this might be some very complicated state, but that does not matter for our purposes

Say that

$$\tilde{a}|\psi\rangle = c_1 a|\psi\rangle + c_2 a^\dagger|\psi\rangle + |\chi\rangle$$

where $|\chi\rangle$ is orthogonal to $a|\psi\rangle$ and $a^\dagger|\psi\rangle$

Such a decomposition can always be made.

Now we have

$$\langle\psi|a^\dagger\tilde{a}|\psi\rangle = 0$$

Since $\langle\psi|a^\dagger a^\dagger|\psi\rangle = 0$ and $\langle\psi|a^\dagger|\chi\rangle = 0$
$\langle\psi|a^\dagger\tilde{a}|\psi\rangle = c_1\langle\psi|a^\dagger a|\psi\rangle$.

So $c_1 = 0$.

Next,

$$\langle \psi | \, a \tilde{a} \, | \psi \rangle = \frac{e^{-\pi \omega_0}}{1 - e^{-2\pi \omega_0}}$$

But

$$\langle \psi | a a | \psi \rangle = 0$$

and

$$\langle \psi | \, a | \psi \rangle = 0$$

so

$$\langle \psi | \, a \tilde{a} \, | \psi \rangle = c_2 \langle \psi | \, a a^{+} \, | \psi \rangle$$

But

$$\langle \psi | a a^{+} | \psi \rangle = \frac{1}{1 - e^{-2\pi \omega_0}}$$

so $c_2 = e^{-\pi \omega_0}$

Also note

$$\langle \psi | \tilde{a}^{\dagger} \tilde{a} | \psi \rangle = \frac{e^{-2\pi\omega_0}}{1 - e^{-2\pi\omega_0}}$$

$$\langle \psi | \tilde{a}^{\dagger} \tilde{a} | \psi \rangle = \underbrace{e^{-2\pi\omega_0}}_{|c_2|^2} \langle \psi | a a^{\dagger} | \psi \rangle + \langle \chi | \chi \rangle .$$

$$= \frac{e^{-2\pi\omega_0}}{1 - e^{-2\pi\omega_0}}$$

So

$$\langle \chi | \chi \rangle = 0$$

We find, putting all together,

$$\boxed{\tilde{a} | \psi \rangle = e^{-\pi\omega_0} a^{\dagger} | \psi \rangle}$$

This is stronger than just the 2-pt correlator of a and $\tilde{a}$.

When we put all these 2-pt correlators together, we find that the action of $\tilde{a}|\psi\rangle$ is parallel to $a^{+}|\psi\rangle$.

Similarly, one can derive

$$\tilde{a}^{+}|\psi\rangle = e^{\pi\omega_0}\, a\,|\psi\rangle .$$

Note different sign.

To summarize. So far, we have explored short-distance entanglement across horizons.

This "entanglement" exists even in the vacuum.

Even though the correlators look thermal, they do not imply any flux at infinity by themselves.

We now move to apply this to black holes.