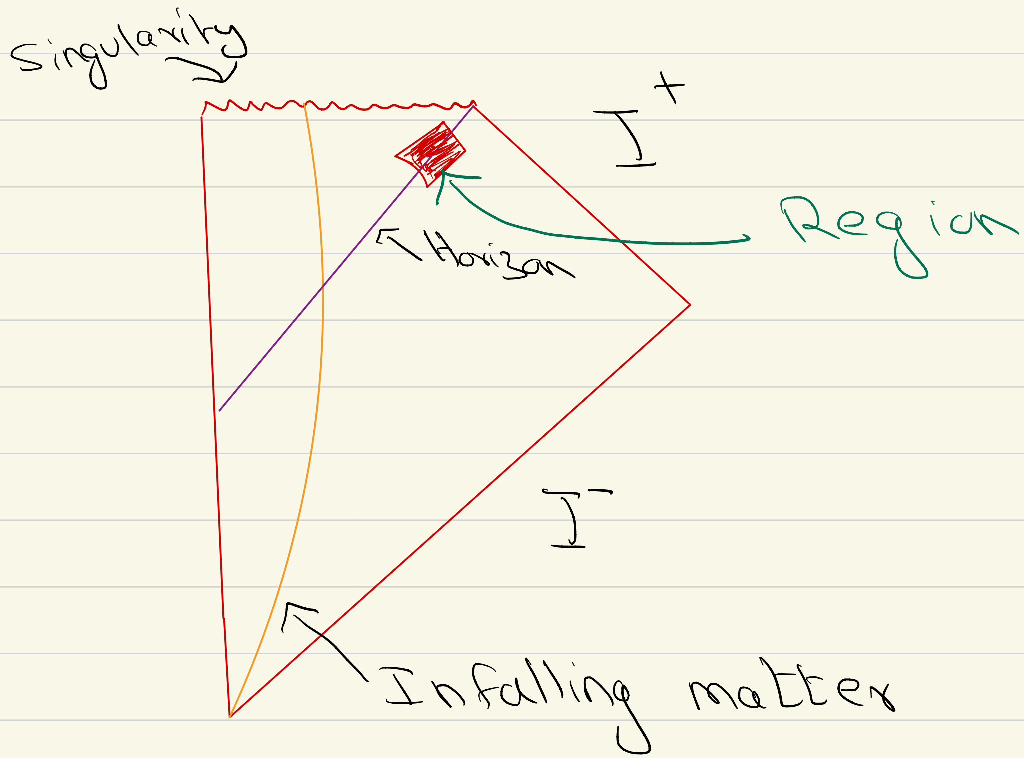
20 Jan 2021. Lecture 3

Black Holes in Flat space.

We are concerned with black holes formed from collapse. On a penrose diagram the process of black hole formation looks like this



The full metric for this scenario is complicated.

This kind of collapse was studied by B. Datt and then by Oppenheimer and Snyder. A simpler kind of collapse is Vaidya collapse.

But at late times, the metric becomes very simple.
We can consider geometries with spherical symmetry. These will be sufficient for us throughout the course

$$ds^2 \xrightarrow[t\to\infty]{} -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2$$

where

$$f(r) = 1 - \frac{\mu}{r^{d-2}}$$

Here μ is related to the "mass" through

$$\mu = 8\pi^{(2-d)/2}\,\Gamma\left(\frac{d}{2}\right) G M / (d-1)$$

The horizon is at

$$r_h = \mu^{1/d-2} \quad \text{where } f(r_h) = 0$$

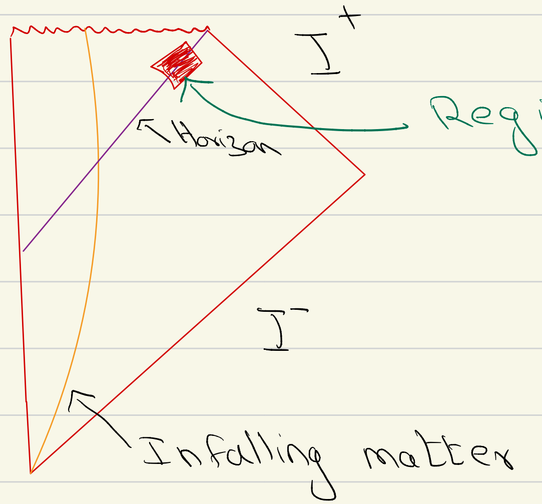
When we say $t \to \infty$ at late times, we mean that

$$t \gg r_h \text{ after collapse}$$

but

$$t \ll t_{evap}$$

At late times, the classical analysis tells us that perturbations die down



Red rectangle looks small but there is a "lot" of time in that rectangle!

It is convenient to move to tortoise coordinates

$$dr_* = \frac{dr}{f(r)}.$$

Let us understand how r_* behaves in different limits.

Near $r \to \infty$, $f(r) \to 1$ so

$$dr_* \to dr$$

Clearly as $r \to \infty$, $r_* \to \infty$

As $r \to r_h$, $f(r) = 2\kappa(r - r_h)$

Here $\kappa = \frac{f'(r_h)}{2}$ is called the surface gravity and will be important in our analysis later.

So near $r \to r_h$, we have

$$dr_* = \frac{dr}{2\kappa(r-r_h)}$$

and so $$r_* = \frac{1}{2\kappa}\log\left[(r-r_h)\underline{2\kappa}\right]$$

↑ choice of const

so $r_* \to -\infty$ near the horizon.

and $$r - r_h = \frac{1}{2\kappa}e^{2\kappa r_*}$$

In terms of r_*, the metric reads

$$ds^2 = f(r)\left[-dt^2 + dr_*^2\right] + r^2 d\Omega^2$$

To cross the horizon, one can move to Kruskal coordinates

$$U = \frac{-1}{\kappa} e^{\kappa(r_* - t)} \quad ; \quad V = \frac{1}{\kappa} e^{\kappa(r_* + t)}$$

Then note that

$$dU = (dt - dr_*) e^{\kappa(r_* - t)}$$

$$dV = (dt + dr_*) e^{\kappa(r_* + t)}$$

$$-dU\, dV = (dr_*^2 - dt^2) e^{2\kappa r_*}$$

But near the horizon we already found that

$$e^{2\kappa r_*} = 2\kappa (r - r_h)$$

The constant of proportionality depends on how we set the "origin" of r_* and we set it above with prescience.

But the important fact is to note

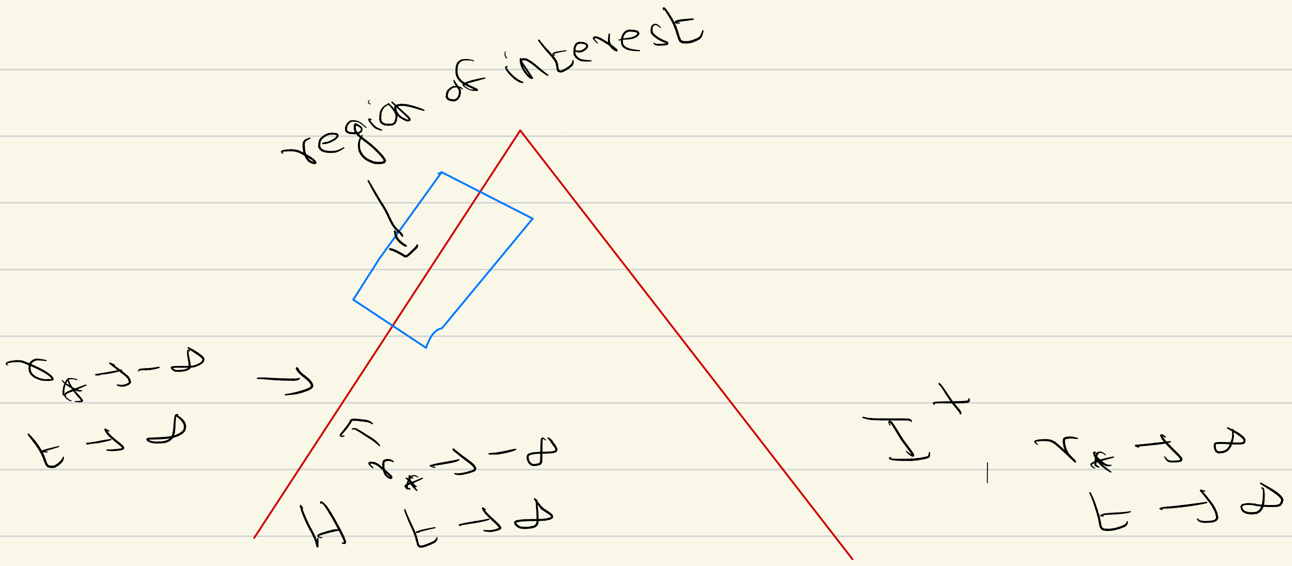
$$ds^2 = -dU\,dV + r^2\,d\Omega_{d-1}^2$$

in the vicinity of the horizon.

So the metric is perfectly smooth at the horizon.

The future horizon is at $U = 0$

and so $t \propto \log \frac{V}{U} \to \infty$



So our region of interest looks as follows

We can set up a second copy of the Schwarzschild coordinates behind the horizon.

There too $r_* \to -\infty$
$t \to \infty$

Now $U = \frac{1}{\kappa} e^{\kappa(r_* - t)}$; $V = \frac{1}{\kappa} e^{\kappa(r_* + t)}$

But notice something curious.

For $r < r_h$, $f(r)$ changes sign!

So t is a spacelike coordinate and r_* is a time coordinate.

In the U, V coordinates there is no issue. We can always use

$$T = V + U$$
$$X = V - U$$

So the fact that space/time "interchange" is just a coordinate artifact.

Propagation of fields

Now we turn to how fields propagate in this geometry

Lets consider a scalar field propagating in this geometry, which satisfies

$$(\Box - m^2)\phi = 0$$

Recall that

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu g^{\mu\nu} \sqrt{-g}\, \partial^\nu$$

In tortoise coordinates

$$\sqrt{-g} = f(r)\, r^{d-1} \sqrt{g_\Omega} \leftarrow \text{From sphere part of metric}$$

$$g^{**} = -g^{tt} = \frac{1}{f(r)}.$$

So the equation becomes

$$\frac{1}{F(r)\, r^{d-1}} \partial_* r^{d-1} \partial_* \phi - \frac{1}{f(r)} \partial_t^2 \phi + \frac{1}{r^2} \nabla_\Omega \phi - m^2 \phi = 0$$

where ∇_Ω is the sphere Laplacian

We can solve this systematically, but the equation simplifies greatly near the horizon.

Near the horizon, we have $F(r) \to 0$

So only two terms remain important! we get

$$\frac{1}{F(r)} \left(\partial_*^2 \phi - \partial_t^2 \phi \right) = 0 \quad \text{as} \quad r \to r_h \,!$$

Note that

1) This is independent of the "angular part"

2) This is independent of the mass

3) If there are additional interactions, those drop out too!

So we find, as a rather robust result, that near the horizon

$$\Phi \to \sum_\omega e^{-i\omega t} \left(A_\omega(\Omega) e^{-i\omega r_*} + B_\omega(\Omega) e^{i\omega r_*} \right) + h.c.$$

arbitrary operator valued fns on the sphere.

A basis of solutions is conventionally chosen as follows. We expand the field also in a basis of spherical harmonics

$$Y_\ell(\Omega)$$

symbol for all angular quntum numbers

Then we choose one solution to be

$$f_{in}(\omega, \ell, r_*)\, e^{-i\omega t}\, Y_\ell(\Omega)$$

and another to be

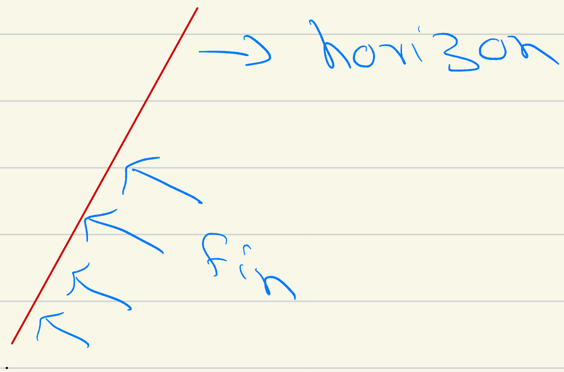
$$f_{out}(\omega, \ell, r_*)\, e^{-i\omega t}\, Y_\ell(\Omega)$$

where

$$f^{in}(\omega, l, r_*) \xrightarrow[r \to r_*^+]{} h_{\omega,l}\, e^{-i\omega r_*}$$

$$f^{out}(\omega, l, r_*) \xrightarrow[r \to r_*^+]{} e^{i\omega r_*} + g_{\omega,l}\, e^{-i\omega r_*}$$

The first solution is denoted by "in" because as t increases r_* must decrease to maintain constant phase



The second solution also has an "outgoing" term at the horizon.

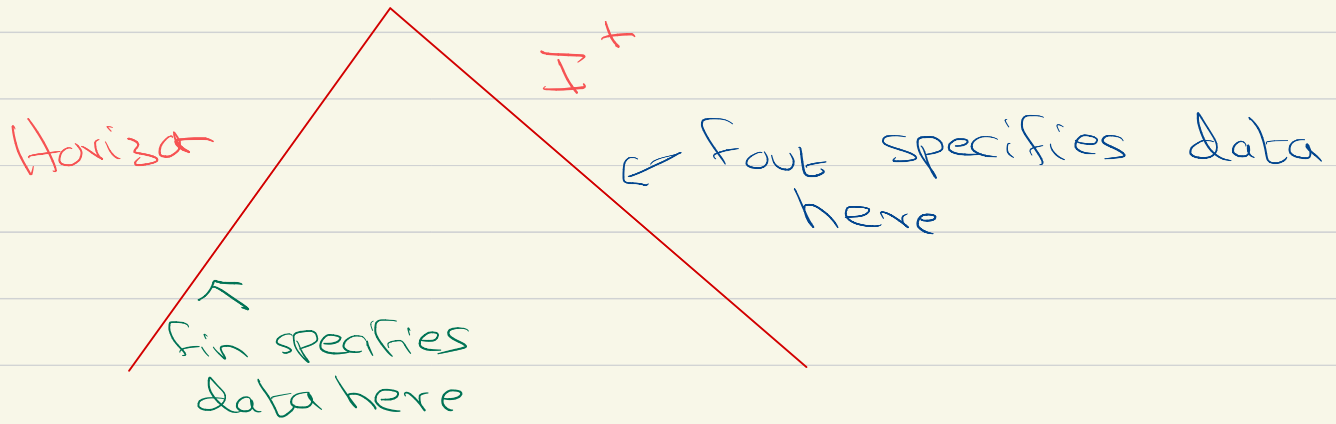
It must also have an ingoing term to remain orthogonal to f_{in} in the Klein Gordon norm.

The constant $g_{\omega,\ell}$ is chosen so that

$$f_{out} = \underbrace{t_{\omega,\ell}}\, r^{(1-d)/2} e^{i\omega r_*} \quad \text{as} \quad r_* \to \infty$$

Assignment question

So another way of thinking about these modes is as follows



$f_{in}\, e^{-i\omega t}$ is regular on the outer horizon since $t \to \infty$ but $r_* \to \infty$ while $r_* + t$ remains finite

But for now, we are only looking at their behaviour near the horizon. We will return to the behavior near ∞ later.

Quantum mechanically, in this basis we can expand

$$\Phi = \sum_{\ell} \int d\omega \left[A_{\omega,\ell}\, f^{out}(\omega,\ell,r_*) + B_{\omega,\ell}\, f^{in}(\omega,\ell,r_*) \right] e^{-i\omega t}\, Y_{\ell}(\Omega) + h.c.$$

we have not normalized $A_{\omega,\ell}$, $B_{\omega,\ell}$ since we do not need the precise normalization.

But
$A_{\omega,\ell}$, $B_{\omega,\ell}$ are annihilation ops
their h.c.'s are creation ops.

We can also quantize fields behind the horizon.

However, now the annihilation operators must multiply $e^{-i\omega r_*}$

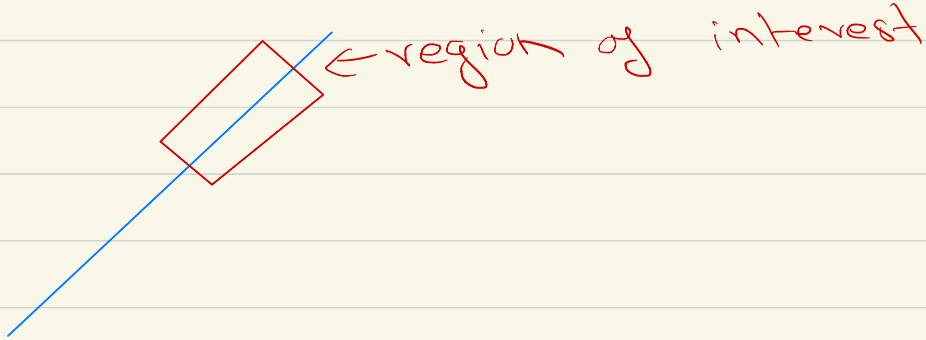
So, behind the horizon we have

Typo in current review. * is missing.

$$\phi = \sum_\ell \int d\omega \left[\tilde{A}_{\omega,\ell} e^{i\omega t} Y_\ell^*(\Omega) + C_{\omega,\ell} e^{-i\omega t} Y_\ell(\Omega) \right] \tilde{f}^{\,out}_{\omega,\ell}(r_*) + h.c.$$

where $\tilde{f}^{\,out}_{\omega,\ell}(r_*) \xrightarrow[r \to r_h^-]{} e^{-i\omega r_*}$

and it is determined away from the horizon by the wave equation.



Note we do not go "deep inside" the horizon.

We remain at $\left|\frac{r_h - r}{r_h}\right| \ll 1$ and away from $t \sim 0$.

Second note that since $e^{-i\omega(t+r_*)}$ is continuous across the horizon

$$C_{\omega,\ell} = A_{\omega,\ell} h_{\omega,\ell} + B_{\omega,\ell} g_{\omega,\ell}$$

But continuity does not fix $\tilde{A}_{\omega,\ell}$ in terms of the modes outside