Lecture 5, SERC SCHOOL, 1 DECEMBER 2015

Today, we will turn to a discussion of QFT in curved space. We will consider scalar fields. The discussion for particles with spin is analogous.

The Lagrangian of a minimally coupled scalar is:

$$L = \frac{1}{2}\int\sqrt{-g}\left[(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} - m^2\phi^2\right]$$

The Euler-Lagrange equations lead to:

$$(\Box + m^2)\,\phi = 0 \Rightarrow \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi + m^2\phi = 0$$

This is a linear p.d.e. and we can find a linear vector space of classical solutions.

To canonically quantize this space, we impose canonical commutation relations.

For this, we need a choice of time. But in a general spacetime, there is no canonical choice of time.

This is what makes the notion of a particle ambiguous.

However, for now we make a choice of time

$$t \equiv x^0.$$

Someone else may make a different choice and we see below how their results will be related.

First the momentum is

$$\pi(t,x) = \frac{\partial S}{\partial(\partial_0 \phi(x))} = g^{0\nu}\sqrt{-g}\,\partial_\nu \phi(t,x)$$

The c.c. relations are then

$$[\phi(t,x), \pi(t,x')] = i\delta(x-x')$$

and, of course

First the momentum is

$$\pi(t,x) \quad \frac{\partial S}{\partial(\partial_0 \phi(x))} = g^{0\nu}\sqrt{-g}\,\partial_\nu \phi(t,x)$$

Returning now to our solutions of the field equation, we expand them in a basis of solutions.

we write

$$\Phi = \sum_i a_i f_i(t,x) + a_i^+ f_i^*(t,x)$$

then we see that

$$[\Phi(t,x), \Pi(t,x')] = [\sum a_i f_i(t,x) + a_i^+ f_i^*(t,x),$$
$$\sqrt{-g}\, g^{0\mu} \partial_\mu \sum a_j f_j(t,x') + a_j^+ f_j^*(t,x')]$$
$$= \sum_{i,j} f_i(t,x) \sqrt{-g}\, g^{0\mu} \partial_\mu f_j^*(t,x') [a_i, a_j^+]$$
$$+ [a_i^+, a_j] f_i^*(t,x) \sqrt{-g}\, g^{0\mu} \partial_\mu f_j(t,x')$$

+ terms involving $[a_i, a_j]$ and $[a_i^+, a_j^+]$

So if we set $[a_i, a_j^\dagger] = \delta_{ij}$ and if the mode-fns satisfy

$$\sum_i \left[f_i(t,x)\, g^{0\mu}\, \partial_\mu f_i^*(t,x') - f_i^*(t,x)\, g^{0\mu}\, \partial_\mu f_i(t,x') \right]$$

$$= \frac{\delta(x-x')}{\sqrt{-g}}$$

then the c.c. relations will be satisfied.

The vacuum is then defined through

$$a_i |\Omega\rangle = 0, \quad \forall i$$

we can then construct states as usual, using

$$a^\dagger_{i_n} \cdots a^\dagger_{i_2} a^\dagger_{i_1} |\Omega\rangle$$

leading to a Fock space.

Now the point is that another observer might find it convenient to use another set of modes. and write the field as:

$$\phi = \sum_i b_i \, g_i(t,x) + b_i^\dagger g_i^*(t,x).$$

Since these are just different bases for the same set of solutions, we will have

$$a_i = \sum \alpha_{ji} b_j + \beta^*_{ji} b_j^+$$

$$a_i^+ = \sum \alpha^*_{ji} b_j^+ + \beta_{ji} b_j$$

We also need

$$\sum (a_i f_i + a_i^+ f_i^+) = \sum_{i,j} (\alpha_{ji} f_i + \beta_{ji} f_i^*) b_j + (\alpha^*_{ji} f_i^* + \beta^*_{ji} f_i) b_j^+$$

this means that

$$g_j = \sum_i \alpha_{ji} f_i + \beta_{ji} f_i^*$$

$$g_j^* = \sum_i \alpha_{ji}^* f_i^* + \beta_{ji}^* f_i$$

Therefore the purely "positive frequency" modes in one basis are a linear combination of positive and negative frequency modes in another basis.

Tutorial: If $[b_i, b_j^\dagger] = \delta_{ij}$, what relation does this imply for α, β?

As a result, the notion of the "vacuum" is not invariant.

$$a_i = \sum \alpha_{ji} b_j + \beta^*_{ji} b_j^+$$

$\Rightarrow$ $a_i |\Omega\rangle = 0$ means

$$\sum_j (\alpha_{ji} b_j + \beta^*_{ji} b_j^+) |\Omega\rangle = 0 \quad \not\Rightarrow b_j |\Omega\rangle = 0$$

Consider the state $b_j |x\rangle = 0$ "b-vacuum"

then we can solve for $|\Omega\rangle$ in terms of $|x\rangle$.

Ansatz:

$$|\Omega\rangle = e^{\frac{1}{2} b_j^+ c_{jk} b_k^+} |x\rangle .$$

Only the symmetric part of C is relevant

$$b_j |\Omega\rangle = \frac{1}{2}\sum_m (C_{jm} b_m^+ + C_{mj} b_m^+) |\Omega\rangle = \sum C_{mj} b_m^+ |\Omega\rangle$$

what we need is

$$\sum \alpha_{ji} b_j |\Omega\rangle = \sum_{j,m} \alpha_{ji} C_{mj} b_m^+ |\Omega\rangle$$

So we need

$$\sum_{j,m} \alpha_{ji} C_{mj} b_m^+ = \sum_m \beta_{mi}^* b_m^+$$

So $\sum_j C_{mj} \alpha_{ji} = \beta_{mi}^* \Rightarrow C = \beta A^{-1}$

So what looks like the vacuum in one frame is a bath of particles excited in all modes in another frame.

The invariant physical quantities are correlators

$$\langle \phi(x_1)\, \phi(x_2) \rangle.$$

These are well-defined quantities that make sense and do not depend on frame except when we try to define $\phi(x)^2$. This is usually defined through a renormalization prescription that may depend on frame.

Now we will work out an example that demonstrates these features. We will just consider flat space in 2 dimensions and quantize it in 2 different ways.

Consider coordinates t, x so that the metric is

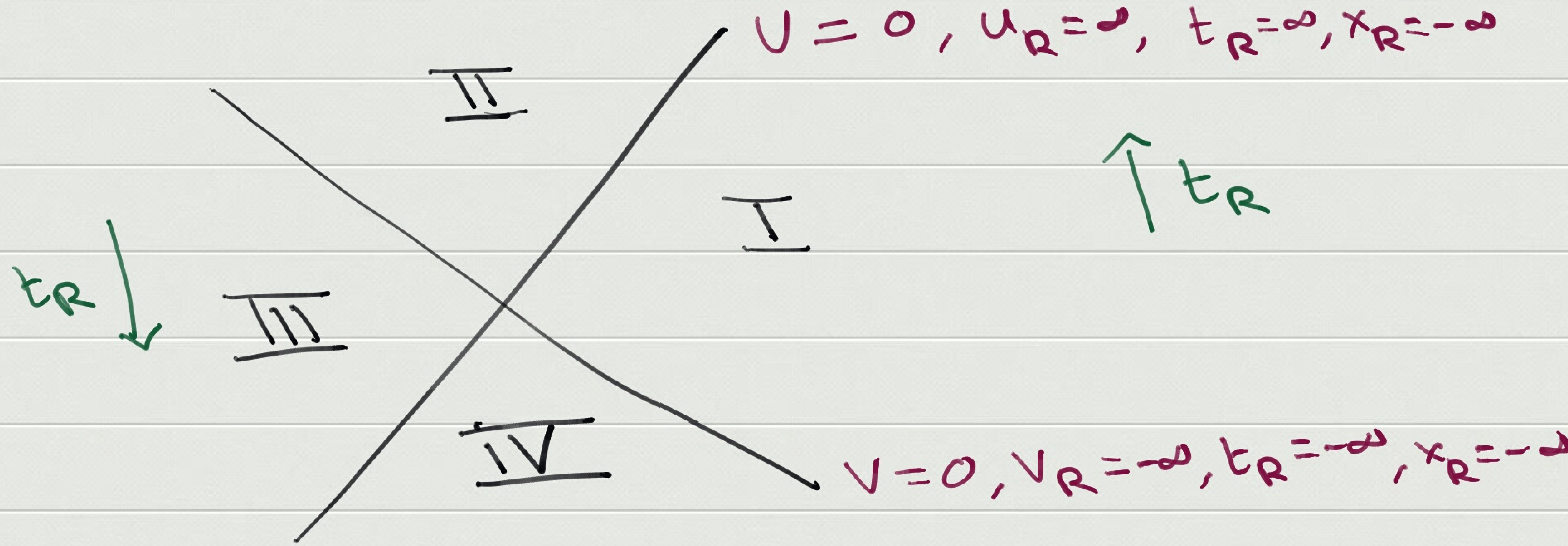
$$ds^2 = dt^2 - dx^2 = dU\,dV \quad ; \quad U = t - x, \quad V = t + x$$

We now write

$$U = -\frac{1}{a} e^{-a u_r}$$

$$V = \frac{1}{a} e^{a v_r}$$

in quadrant I.



Notice that $dU\,dV = \frac{1}{a^2}\, e^{a(v_R-u_R)}\, du_R\, dv_R\,.$

$$= \frac{1}{a^2}\, e^{2ax_R}\left[dt_R^2 - dx_R^2\right]. \quad , \quad u_R = t_R - x_R$$

$$v_R = t_R + x_R$$

The x_R, t_R coordinates are called Rindler coordinates.

Tutorial: Show that x_R = constant is the worldline of a uniformly accelerated observer.

Notice that

$$g^{x_R x_R} = e^{-2a x_R} a^2 = g^{t_R t_R}$$

but

$$\sqrt{-g} = \frac{1}{a^2} e^{2a x_R}$$

So the wave equation in these coordinates is again

$$\left(\frac{\partial^2}{\partial t_R^2} - \frac{\partial^2}{\partial x_R^2}\right)\phi = 0$$

$$\Rightarrow \phi_{I} = \int \frac{d\omega}{\sqrt{\omega}}\left[e^{-i\omega u_R} a_\omega + e^{-i\omega v_R} b_\omega + h.c\right]$$

Similarly, in region $\overline{\underline{III}}$ we introduce a second set of Rindler coordinates through

$$U = \frac{1}{a} e^{-a u_R} \quad ; \quad V = \frac{-1}{a} e^{a v_R}$$

The field here looks like

Note +ve signs because of direction of t_R

$$\Phi_{\text{III}} = \int \frac{d\omega}{\sqrt{\omega}} \left[\tilde{a}_\omega e^{i\omega U_R} + \tilde{b}_\omega e^{i\omega V_R} + h.c. \right]$$

If we think of the slice at $T=0$, we can then write

$$\Phi = \int \frac{d\omega}{\sqrt{\omega}} \Big[\tilde{a}_\omega U_L(u_r) + \tilde{b}_\omega V_L(V_R) + a_\omega U_R(u_R) + b_\omega V_R(V_R) + h.c. \Big]$$

where

$$V_L(u_R) = e^{i\omega u_R}, \quad \text{Region III (on left)}$$
$$= 0, \quad \text{region I (on right)}$$

$$V_R(u_R) = \begin{cases} 0, & \text{Region III} \\ e^{-i\omega u_R}, & \text{Region I} \end{cases}$$

$$V_L(v_R) = e^{i\omega v_R}, \quad \text{III}$$
$$= 0, \quad \text{I}$$

$$V_R(v_R) = \begin{cases} 0, & \text{III} \\ e^{-i\omega v_R} & \text{I} \end{cases}$$

On the other hand, we can also write

$$\Phi = \int \frac{d\omega}{\sqrt{\omega}} \left[c_\omega e^{-i\omega(t-x)} + d_\omega e^{-i\omega(t+x)} + h.c. \right]$$

ordinary Minkowski coordinates

What are the Bogoliubov coefficients between these two expansions?