QABA, Lecture 2, QFT in curved space, 19 Aug 2016

Today, we will review some basic tools involving quantum fields in curved spacetime.
First, it is useful to recollect some properties of QFT in flat spacetime, and then we will see which ones carry over.
In flat space, consider

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\nu \phi)\eta^{\mu\nu} - \frac{m^2 \phi^2}{2}$$

with $S = \int \mathcal{L}\, d^4x$

The equation of motion is

$$(\Box + m^2)\phi = 0.$$

This is usually solved by writing

$$\phi = \int \frac{d^3 k}{\sqrt{2\omega_k}\,(2\pi)^{3/2}} \left(a_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + h.c. \right)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2} > 0$

The canonical commutation relations, with

$$\pi(x) = \frac{\partial L}{\partial \partial_0 \phi} = \partial_0 \phi$$

read

$$[\phi(t,x), \pi(t,x')] = i\,\delta^3(x-x')$$

In terms of a_k, these translate into

$$[a_k, a_{k'}^{\dagger}] = \delta(k-k')$$

We now define a distinguished state, the vacuum by

$$a_k |\Omega\rangle = 0 \quad \forall\, k$$

we then define states on top of the vacuum by

$$|n_{k_1} \dots n_{k_n}\rangle = \prod_k \frac{(a_k^{\dagger})^{n_k}}{\sqrt{n_k!}} |\Omega\rangle.$$

Also, the key observables are vacuum correlators,

$$\langle 0 | \phi(x_1) \dots \phi(x_n) | 0 \rangle$$

As we know, it is possible to choose several Lorentz frames. However, under a Lorentz transformation, the vacuum is invariant.

$$U_\Lambda |\Omega\rangle = |\Omega\rangle .$$

Moreover one-particle states transform through

$$U_\Lambda |k\rangle = |\Lambda(k)\rangle$$

where $\Lambda(k)$ is the transformed momentum. The transformed state is also a one-particle state.

We now turn to QFT in curved space. The idea here is that there is some relatively fixed background spacetime and we are interested in fluctuations of matter fields about this spacetime. We can also consider gravity waves on this background.

QFT in curved space is a first step to understanding quantum effects in gravity.

The essential formulation of QFT goes through as previously in curved space except

a) the mode-functions are not plane waves
b) not all observers agree on the vacuum
c) not all observers agree on the number of particles

Fact (a) is simply a result of the metric not being flat and the coupling of curvature to the fluctuations.

Facts (b) and (c) result from there being no canonical choice of time

We will first consider some general formalism and then see (a), (b), (c) in concrete examples.

We start by considering a minimally coupled scalar

$$L = \frac{1}{2} \int \sqrt{-g} \left[(\partial_\mu \phi)(\partial_\nu \phi) g^{\mu\nu} - m^2 \phi^2 \right] \quad ; \quad \sqrt{-g} = \det(g_{\mu\nu})$$

This is not the unique generalization of the flat space action. eg. $L \to L + \xi \int \sqrt{-g} R \phi^2$ has the same flat space limit. We consider minimal coupling for simplicity

The Euler-Lagrange equations lead to:

$$(\Box + m^2)\phi = 0 \Rightarrow \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi + m^2 \phi = 0$$

This is a linear p.d.e. and we can find a linear vector space of classical solutions.
To canonically quantize this space, we impose canonical commutation relations.
For this, we need a choice of time. But in a general spacetime, there is no canonical choice of time.
This is what makes the notion of a particle ambiguous, as we mentioned earlier.
However, for now we make a choice of time

$t \equiv x^0$. in our favourite coordinate system.

Someone else may make a different choice and we see below how their results will be related.

First the momentum is

$$\Pi(t,x) = \frac{\partial S}{\partial(\partial_0 \phi(x))} = g^{0\nu}\sqrt{-g}\,\partial_\nu \phi(t,x)$$

The c.c. relations are then

$$[\phi(t,x), \Pi(t,x')] = i\delta(x-x')$$

Despite appearances, these relations are covariant. To see this, it is useful to write the metric using a 3+1 split as

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Then, one can check that

$$\sqrt{-g} = N\sqrt{h}$$

and also

$$g^{00} = \frac{1}{N^2}\;; \quad g^{ij} = -h^{ij} + \frac{N^i N^j}{N^2}\;; \quad g^{0i} = -\frac{N^i}{N^2}$$

and also that the vector

$$n = \frac{1}{N}\left(\frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \right)$$

has unit norm: $g_{\mu\nu} n^\mu n^\nu = -1$;

and we can write

$$g^{0\mu} = \frac{1}{N} n^\mu$$

and so the commutation relations can be written as

$$\left[\phi(t,x),\ n^\mu \partial_\mu \phi(t,x')\right] = \frac{i\,\delta(x-x')}{\sqrt{h}}$$

Now someone else may choose a different slicing of the spacetime. But then that person's quantities are related by diffeomorphisms. Everything matches since

$$\delta(\tilde{x}-\tilde{x}') = \delta(x-x') / \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \leftarrow \text{only spatial}$$

$$\sqrt{\tilde{h}} = \det(\partial x^i / \partial \tilde{x}^j)\ \sqrt{h}$$

we write

$$\phi = \sum_i a_i f_i(t,x) + a_i^{+} f_i^{*}(t,x)$$

then we see that

$$[\Phi(t,x), \Pi(t,x')] = [\sum a_i f_i (t,x) + a_i^{+} f_i^{*} (t,x),$$

$$\sqrt{-g}\, g^{0\mu} \partial_\mu \sum a_j f_j (t,x') + a_j^{+} f_j^{*} (t,x')]$$

$$= \sum_{i,j} f_i (t,x) \sqrt{-g}\, g^{0\mu} \partial_\mu f_j^{*} (t,x') [a_i, a_j^{+}]$$

$$+ [a_i^{+}, a_j] f_i^{*} (t,x) \sqrt{-g}\, g^{0\mu} \partial_\mu f_j (t,x')$$

+ terms involving $[a_i, a_j]$ and $[a_i^{+}, a_j^{+}]$

So if we set $[a_i, a_j^\dagger] = \delta_{ij}$ and if the mode-fns satisfy

$$\sum_i \left[f_i(t,x)\, g^{0\mu}\, \partial_\mu f_i^*(t,x') - f_i^*(t,x)\, g^{0\mu}\, \partial_\mu f_i(t,x') \right]$$
$$= \frac{\delta(x-x')}{\sqrt{-g}}$$

then the c.c. relations will be satisfied.

we can then construct states as usual, using

$$a^{\dagger}_{i_n} \cdots \, a^{\dagger}_{i_2} a^{\dagger}_{i_1} |\Omega\rangle$$

leading to a Fock space.

Now the point is that another observer might find it convenient to use another set of modes. and write the field as:

$$\phi = \sum_i b_i \, g_i \, (t,x) + b_i^{\dagger} \, g_i^{*} \, (t,x).$$

Since these are just different bases for the same set of solutions, we will have

$$a_i = \sum \alpha_{ji} b_j + \beta^*_{ji} b_j^+$$

$$a_i^+ = \sum \alpha^*_{ji} b_j^+ + \beta_{ji} b_j$$

We also need

$$\sum (a_i f_i + a_i^+ f_i^*) = \sum_{i,j} (\alpha_{ji} f_i + \beta_{ji} f_i^*) b_j + (\alpha^*_{ji} f_i^* + \beta^*_{ji} f_i) b_j^+$$

this means that

$$g_j = \sum_i \alpha_{ji} f_i + \beta_{ji} f_i^*$$

$$g_j^* = \sum_i \alpha_{ji}^* f_i^* + \beta_{ji}^* f_i$$

Therefore the purely "positive frequency" modes in one basis are a linear combination of positive and negative frequency modes in another basis.

We also need

$$[b_i , b_j^+] = \delta_{ij}$$

Using the relation between a & b

$$[a_i , a_j^+] = [\alpha_{ti} b_t + \beta^*_{ti} b_t^+ ,$$

$$\alpha^*_{qj} b_q^+ + \beta_{qj} b_q]$$

$$= \alpha_{ti} \alpha^*_{tj} - \beta^*_{ti} \beta_{tj}$$

$$= \delta_{ij}$$

Since frequencies mix, the notion of the "vacuum" is not invariant.

$$a_i = \sum \alpha_{ji} b_j + \beta^*_{ji} b_j^+$$

$\Rightarrow$ $a_i |\Omega\rangle = 0$ means

$$\sum_j (\alpha_{ji} b_j + \beta^*_{ji} b_j^+) |\Omega\rangle = 0 \quad \not\Rightarrow b_j |\Omega\rangle = 0$$

Consider the state $b_j |x\rangle = 0$ "b-vacuum"

then we can solve for $|\Omega\rangle$ in terms of $|x\rangle$.

Ansatz:

$$|\Omega\rangle = e^{\frac{1}{2} b_j^+ C_{jk} b_k^+} |x\rangle$$

$\Leftarrow$ Motivate using coherent states

We find

$$b_j |\Omega\rangle = \frac{1}{2}\sum_m (C_{jm} b_m^+ + C_{mj} b_m^+) |\Omega\rangle = \sum C_{mj} b_m^+ |\Omega\rangle$$

and so

$$\sum \alpha_{ji} b_j |\Omega\rangle = \sum_{j,m} \alpha_{ji} C_{mj} b_m^+ |\Omega\rangle$$

The equation is

$$\sum_{j,m} \alpha_{ji} C_{mj} b_m^+ = -\sum_m \beta^*_{mi} b_m^+$$

or $\sum_j C_{mj} \alpha_{ji} = -\beta^*_{mi} \Rightarrow C_{mj} = -\sum_i \beta^*_{mi} \gamma_{ij}$

where

$$\sum_i \alpha_{ji} \gamma_{ik} = \delta_{jk}$$

So what looks like the vacuum in one frame is a bath of particles excited in all modes in another frame.

The invariant physical quantities are correlators

$$\langle \phi(x_1) \phi(x_2) \rangle.$$

These are well-defined quantities that make sense and do not depend on frame