Quantum Aspects of Black Holes — Lecture 10

We now turn to more general black hole solutions.

Given the Schwarzschild black hole, we can simply throw some charge into it. This gives rise to the Reissner-Nordstrom metric

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} + r^2 d\Omega^2$$

$$A_\mu = -\frac{Q}{r}\,(dt_\mu)$$

It is not too difficult to check that this obeys the eom.

First note that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{rt} = -F_{tr} = \frac{Q}{r^2}$$

Second recall that

$$S = \frac{-1}{8\pi} \int \sqrt{-g} \; F_{\mu\nu} F^{\mu\nu}$$

and so $T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{-1}{4\pi} F_{\mu\rho} F_{\sigma\nu} g^{\rho\sigma} + \frac{1}{16\pi} g_{\mu\nu} (F_{\mu\nu} F^{\mu\nu})$

Some properties of this solution are similar to the Schwarzschild solution. The hypersurface orthogonal killing field is $\frac{\partial}{\partial t}$

The interesting new element is that $g_{tt} \to 0$ at two points. We have

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) = 0 \implies r^2 - 2Mr + Q^2 = 0$$

$$\implies r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

The larger value, r_+, is called the outer horizon.

We will come to the inner horizon shortly but notice that for $M^2 = Q^2$, the two horizons coincide. For $Q^2 > M^2$, the solution has a naked singularity.

So

$$M^2 = Q^2$$

is called the extremal limit of this solution.

It is interesting to compute

$$\left|\frac{Q}{M}\right|$$

for an electron!

Recall that $\frac{Q^2}{4\pi\varepsilon_0 r^2}$ and $\frac{GM^2}{r^2}$ have the same units

so

$$\frac{Q}{M} \sim \sqrt{\frac{F_{electric}}{F_{grav}}}$$ between two electrons

$$F_{electric} = \underbrace{9 \times 10^{9}}_{\text{coulomb const}} \times \frac{(1.6 \times 10^{-19})^2}{r^2} \, N \sim \frac{23 \times 10^{-29}}{r^2} \, N$$

$$F_{grav} = 6.67 \times 10^{-11} \times \frac{(9.1 \times 10^{-31})^2}{r^2} \, N \sim \frac{55.2 \times 10^{-71}}{r^2} \, N$$

$$\sqrt{\frac{F_{elect}}{F_{grav}}} \sim 10^{20} \, !$$

Q: What happens if you drop an electron into an extremal B.H. ?

In fact, one cannot drop the electron. If $q > M$, the electron will be repelled, and not fall in. To throw a particle in, one has to give it enough energy that $E > q$.

Note that there is no difficulty in emitting such particles in the extremal case.

Now lets consider the near horizon geometry. As usual we introduce tortoise coordinates. We want

$$dr_*^2 = \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^2} = \frac{dr^2}{\left(1 - \frac{r_+}{r}\right)^2 \left(1 - \frac{r_-}{r}\right)^2}$$

or

$$r_* = r - \frac{r_-^2}{r_+ - r_-} \ln\left(\frac{r - r_-}{r_-}\right) + \frac{r_+^2}{r_+ - r_-} \ln\left(\frac{r - r_+}{r_+}\right)$$

Near $r = r_+$, we can again introduce global coordinates

$$U = -e^{\alpha(r_* - t)} \quad ; \quad V = e^{\alpha(r_* + t)}$$

We find again that

$$dU\,dV = \alpha^2 e^{2\alpha r_*} (dt^2 - dr_*^2)$$

near $r = r_+$, we have

$$r_* \sim \frac{r_+^2}{(r_+ - r_-)} \ln\left(\frac{r - r_+}{r_+}\right).$$

or

$$e^{2\alpha r_*} \sim (r - r_+)^{2\alpha r_+^2/(r_+ - r_-)}$$

If we choose $\alpha = \dfrac{r_+ - r_-}{2 r_+^2}$

the metric will become regular and proportional to $dU\,dV$ near the outer horizon.

This is an important number. Recall that in our discussion of Rindler <-> Minkowski coefficients when

$$U_M = -e^{a(x_R - t_R)}$$
$$V_M = e^{a(x_R + t_R)}$$

the temperature perceived by the Rindler observer was $a/2\pi$

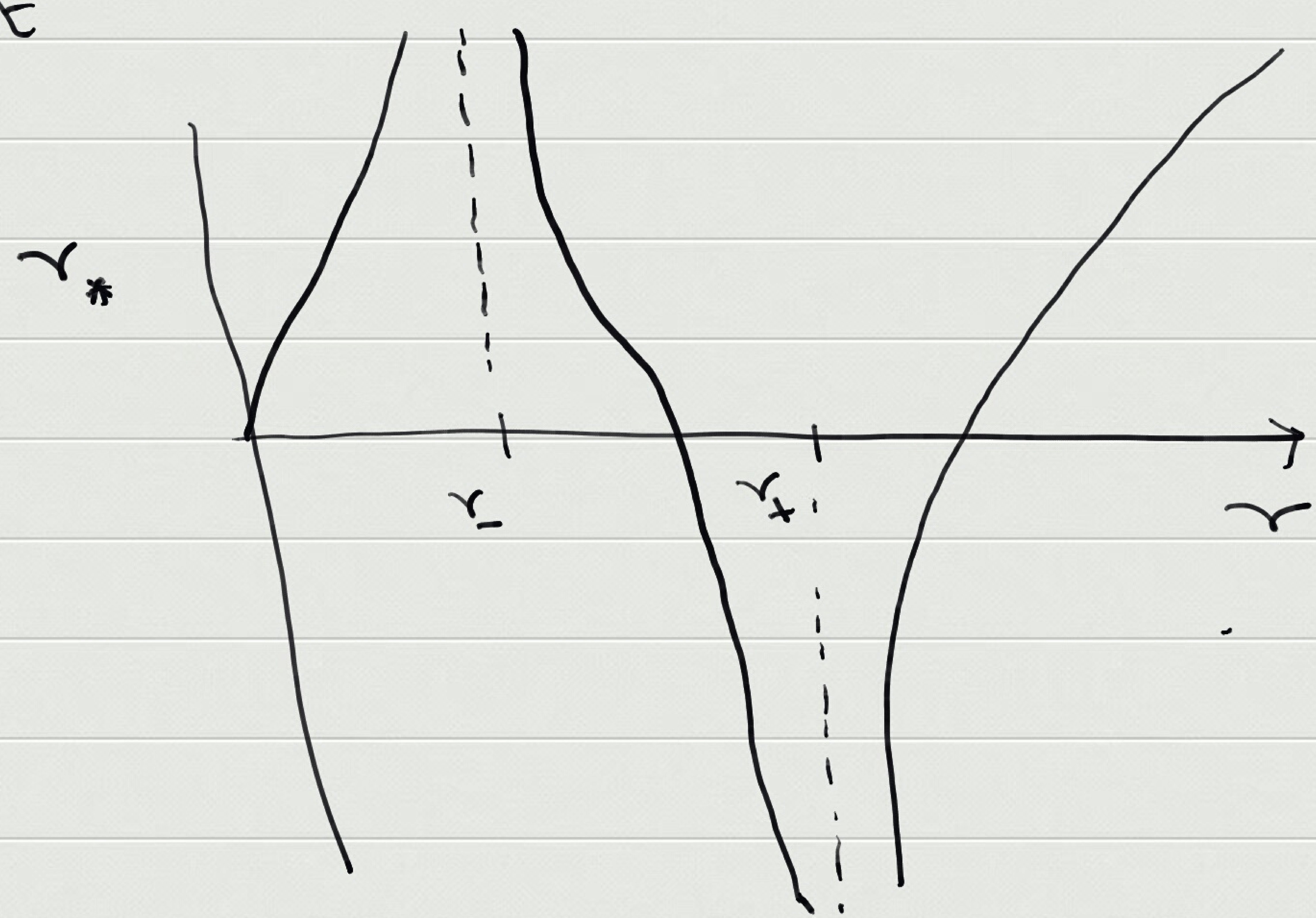
The calculation above hints that the temperature of a R-N B.H. is

$$T = \frac{(r_+ - r_-)}{4\pi r_+^2}$$

and we will see that this is correct when we study Hawking radiation.

We now turn to the maximal extension of this geometry.

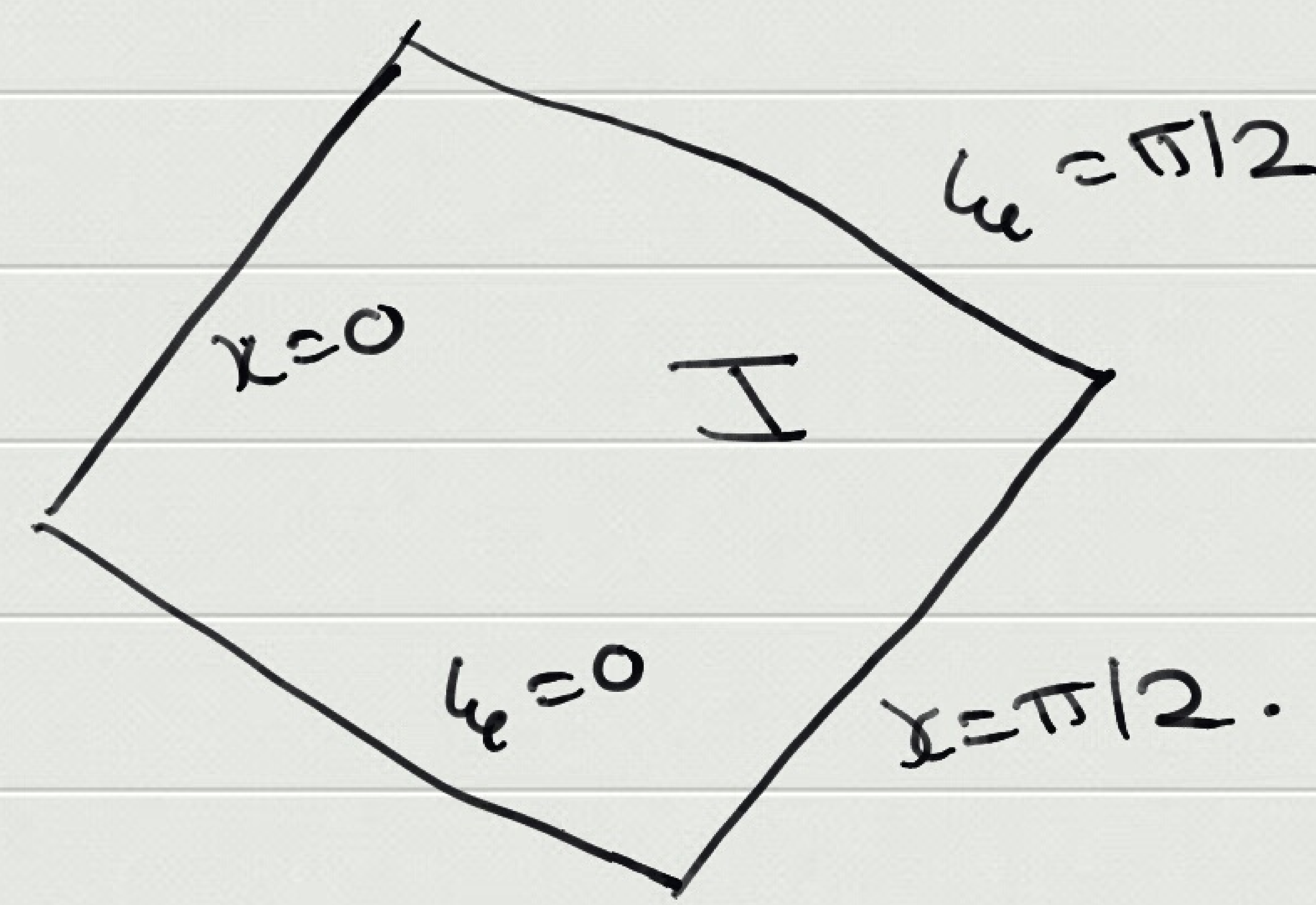
Here the graph of r_* vs r looks somewhat different



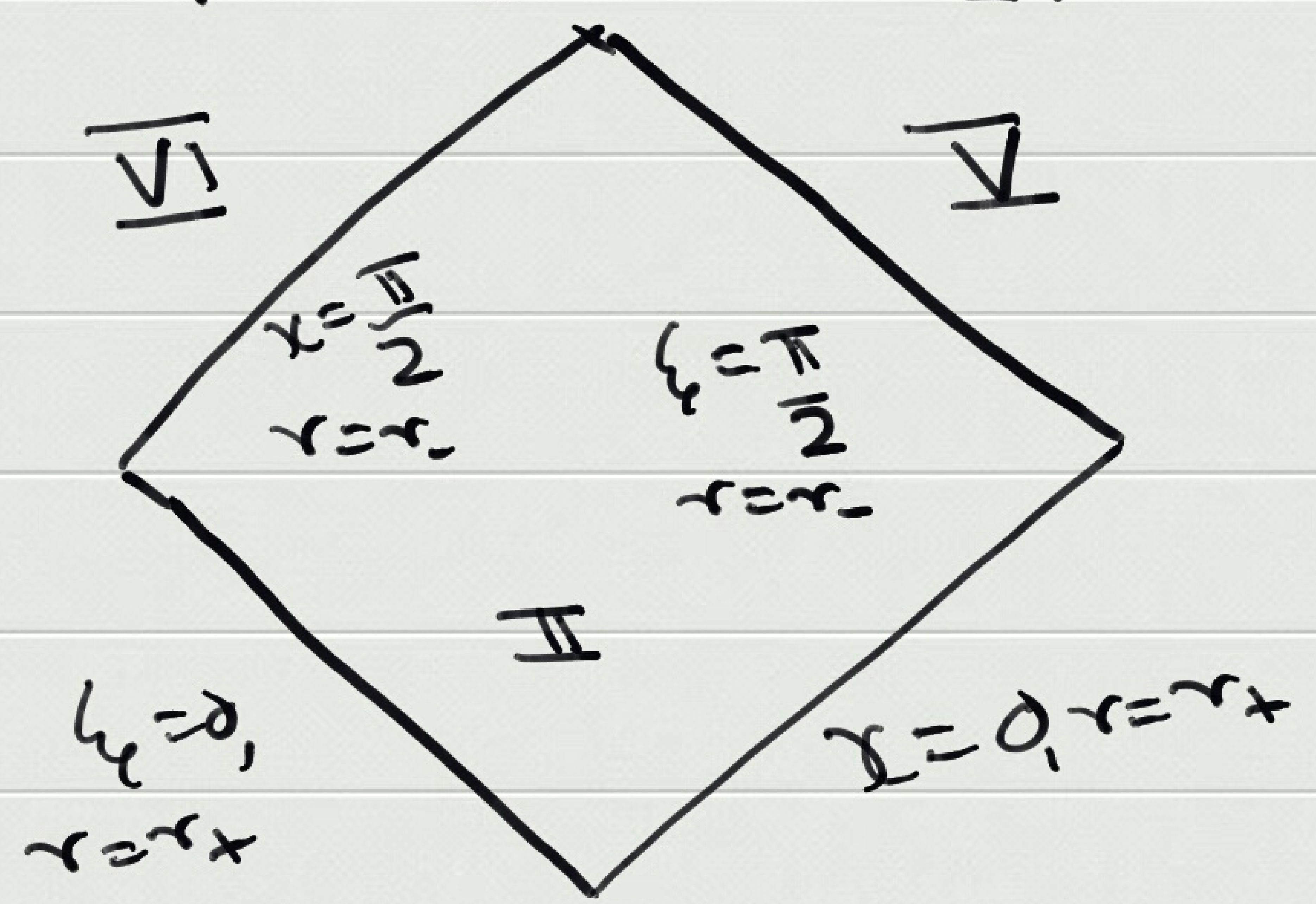
corresponding to this, let us introduce coordinates

$$\chi = -\tan^{-1} U \quad ; \quad \psi = \tan^{-1} V$$

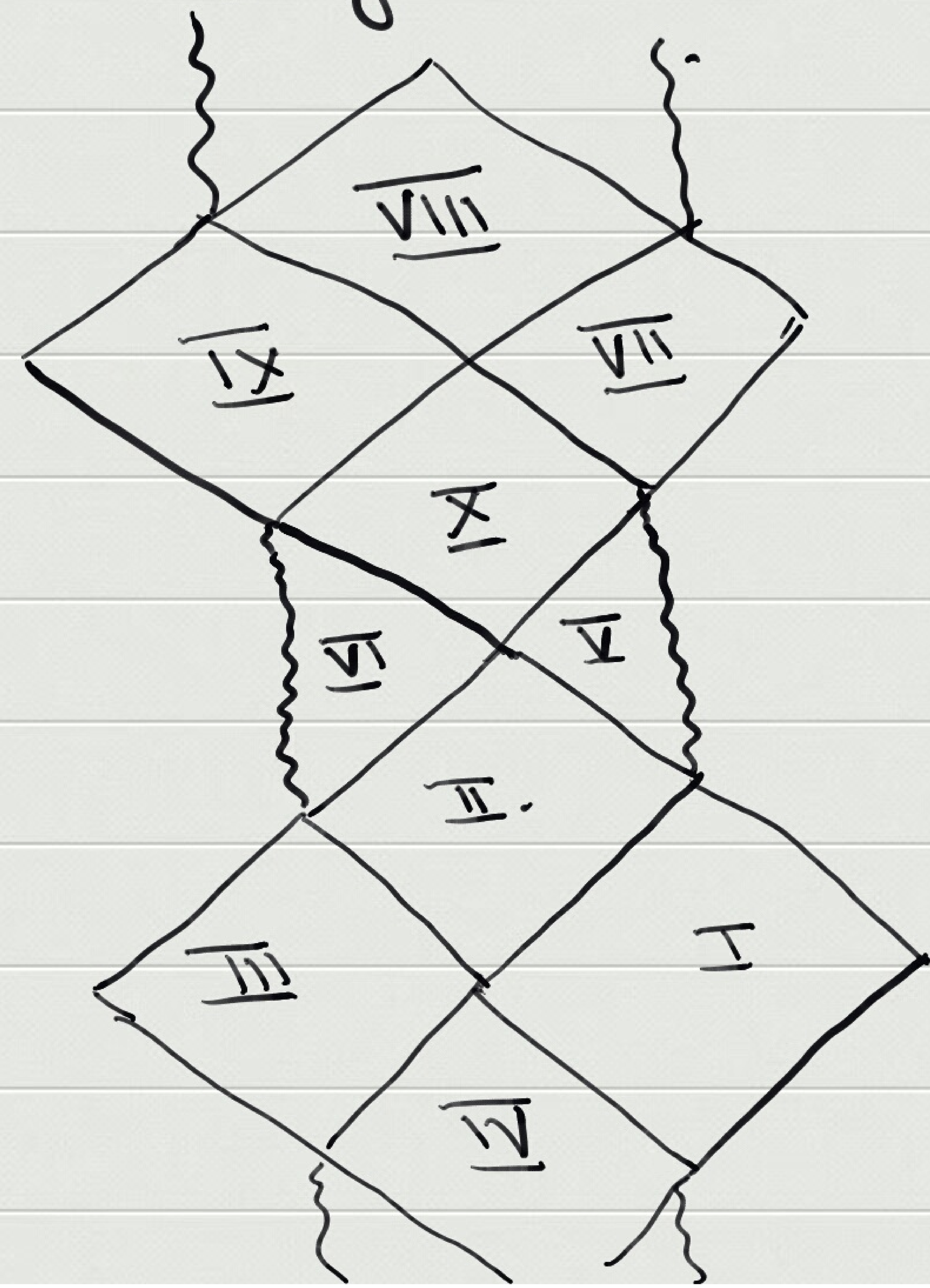
as usual we see that $r_* \to -\infty$ corresponds to $UV = 0$
so the diagram for region I covered vy the original
patch is



Now we extend past $V=0$. As usual lines of constant r_* are hyperboloids. But now note that $UV=1$, is not a singular point, because this corresponds to some value of r intermediate between r_+ and r_-. We can now go past $\xi=\frac{\pi}{2}$, and $\chi=\frac{\pi}{2}$ corresponding to the part of $r<r_-$ into regions V and VI.



in regions $\overline{\underline{V}}$ and $\overline{\underline{VI}}$, we eventually hit the singularity so the full diagram looks like

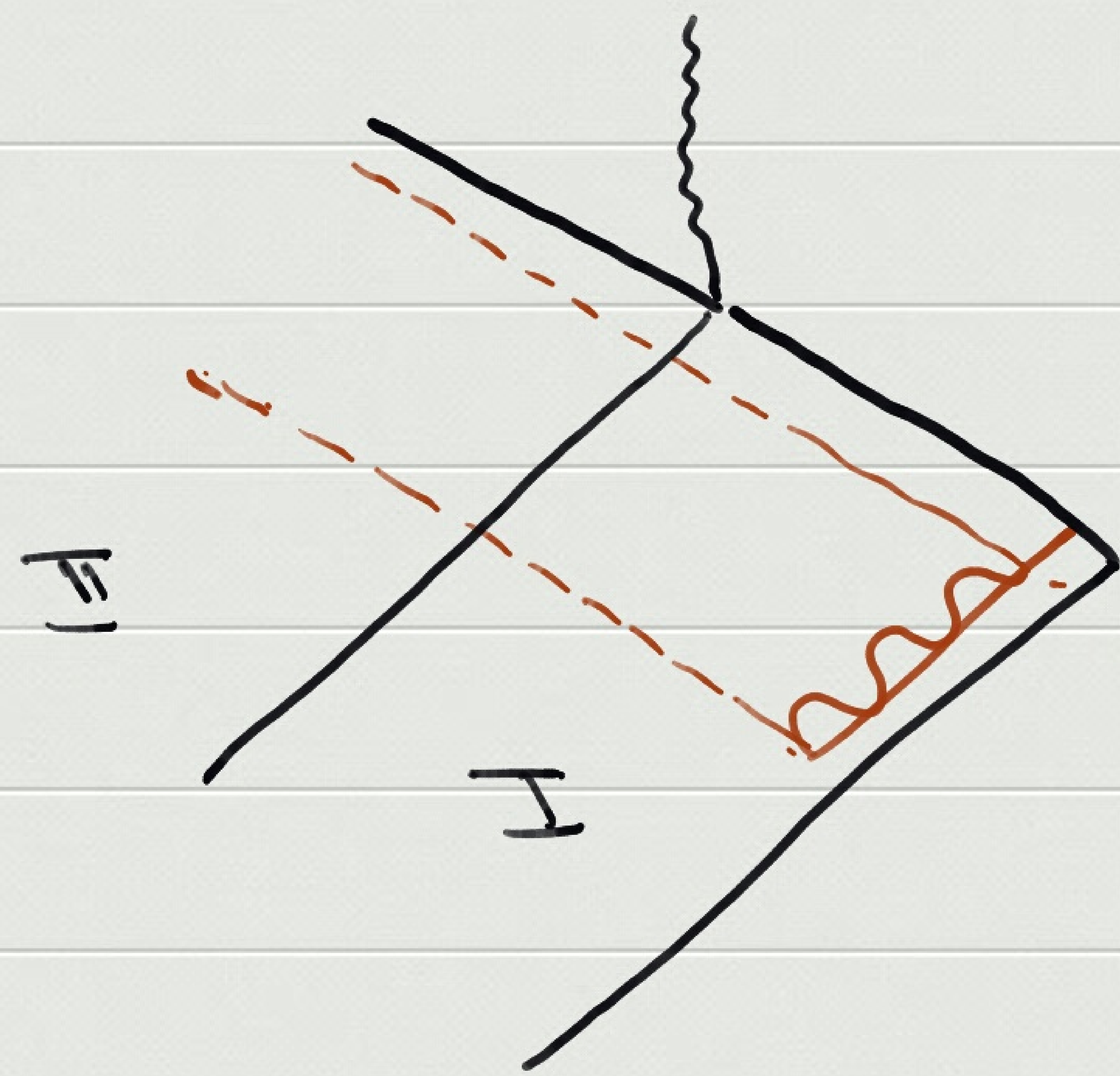


however this diagram is also somewhat fake.
There are 2 reasons:

a) as we saw in the Oppenheimer-Snyder collapse, a collapsing star covers most of the "left" of the diagram.

b) the 'inner horizon' is actually unstable to small perturbations.

Consider a plane wave solution. Since there is infinite affine distance from $\psi = 0$ to $\pi/2$, the wave has infinite oscillations in this interval.



But an infalling observer crosses this infinite interval in finite proper time between r_+ to r_-. → infinite blue shift!

We can see this in more detail. Consider

$$V_{as} = t + r_*$$

and consider a solution, where we specify data on $\mathcal{I}^-$ to be

$$\phi_{\mathcal{I}^-} \propto e^{-i\omega V_{as}}$$

Now consider an infalling observer. This observer perceives this wave to have frequency ω_{obs}:

$$\omega_{obs} = \omega \frac{dV_{as}}{d\tau}$$

Near the inner horizon, we have $v_{as} \to \infty$ and we make the metric well behaved by changing to coordinates $V = e^{-\gamma v_{as}}$. In these coordinates clearly

$$\frac{dV}{d\tau}$$

is finite.

But $\dfrac{dv_{as}}{dV} \to \infty$. So $\dfrac{dv_{as}}{d\tau} = \dfrac{dv_{as}}{dV} \cdot \dfrac{dV}{d\tau} \to \infty$

so $\omega_{as} \to \infty$.

So a more realistic diagram is