# Quantum Aspects of Black Holes, Lecture 13

In the previous lecture, we established the Raychaudhuri eqn.

We considered a congruence of geodesics with tangent vector given by $\xi^a$. If $\eta^a$ was the "deviation" from one element of the congruence to another, we defined

$$\xi^a \nabla_a \eta^b = B_a{}^b \eta^a$$

We decomposed

$$B_{ab} = \frac{1}{2}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$$

the term of greatest interest to us is $\theta \rightarrow$ the expansion.

For congruences that were hypersurface orthogonal we proved that

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$$

so

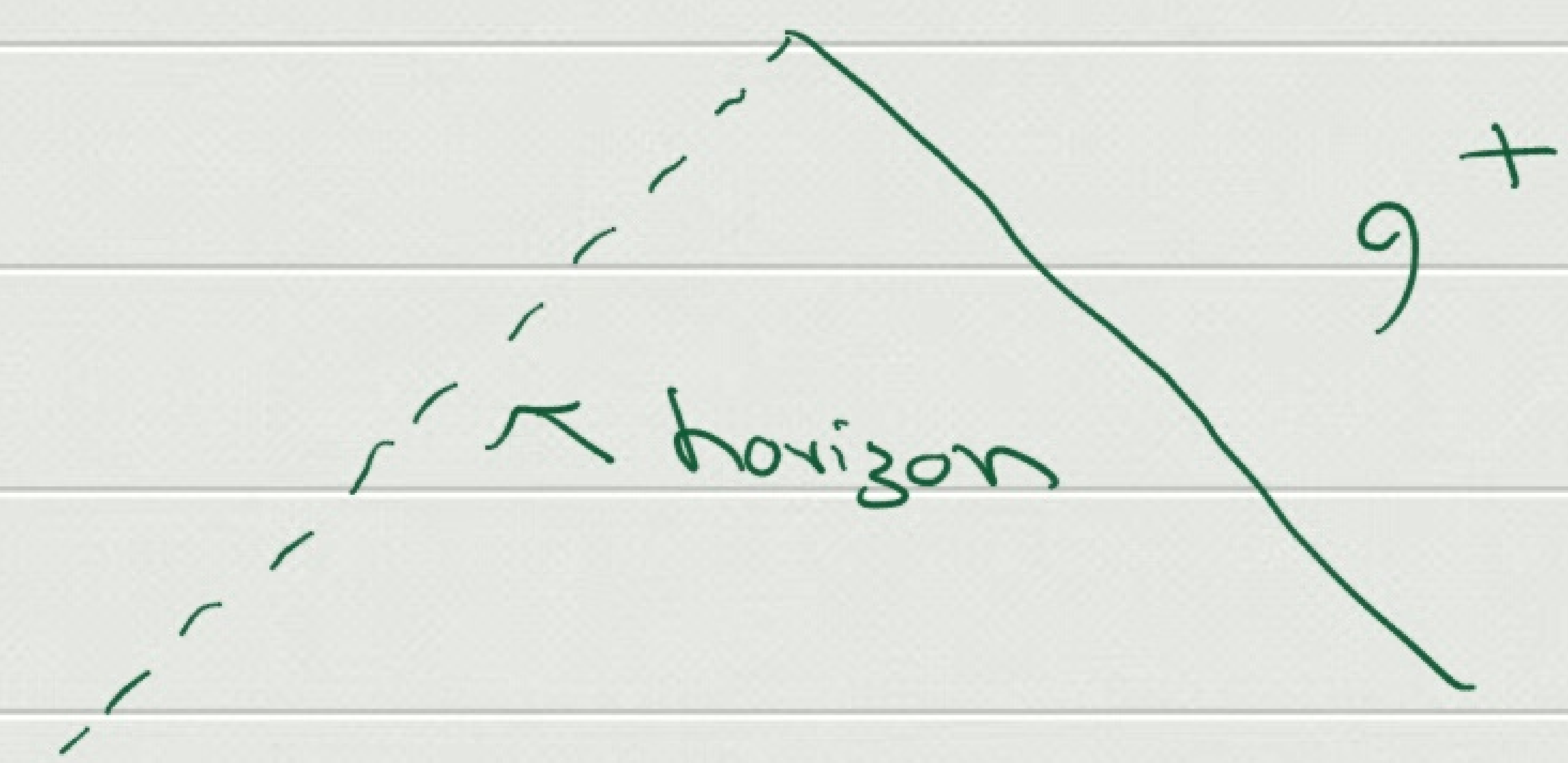
$$\frac{1}{\theta_f} - \frac{1}{\theta_i} \leq \frac{1}{2}(\lambda_f - \lambda_i)$$

If $\theta_i$ becomes negative at any point then in finite affine parameter, $\theta_f \to -\infty$.

We can now turn to horizons and the area theorem.

Let us recall what the horizon is. A black hole is a region from where a particle cannot escape to asymptotic infinity in the future.

So the horizon is the boundary of the causal past of $\mathscr{I}^+$.

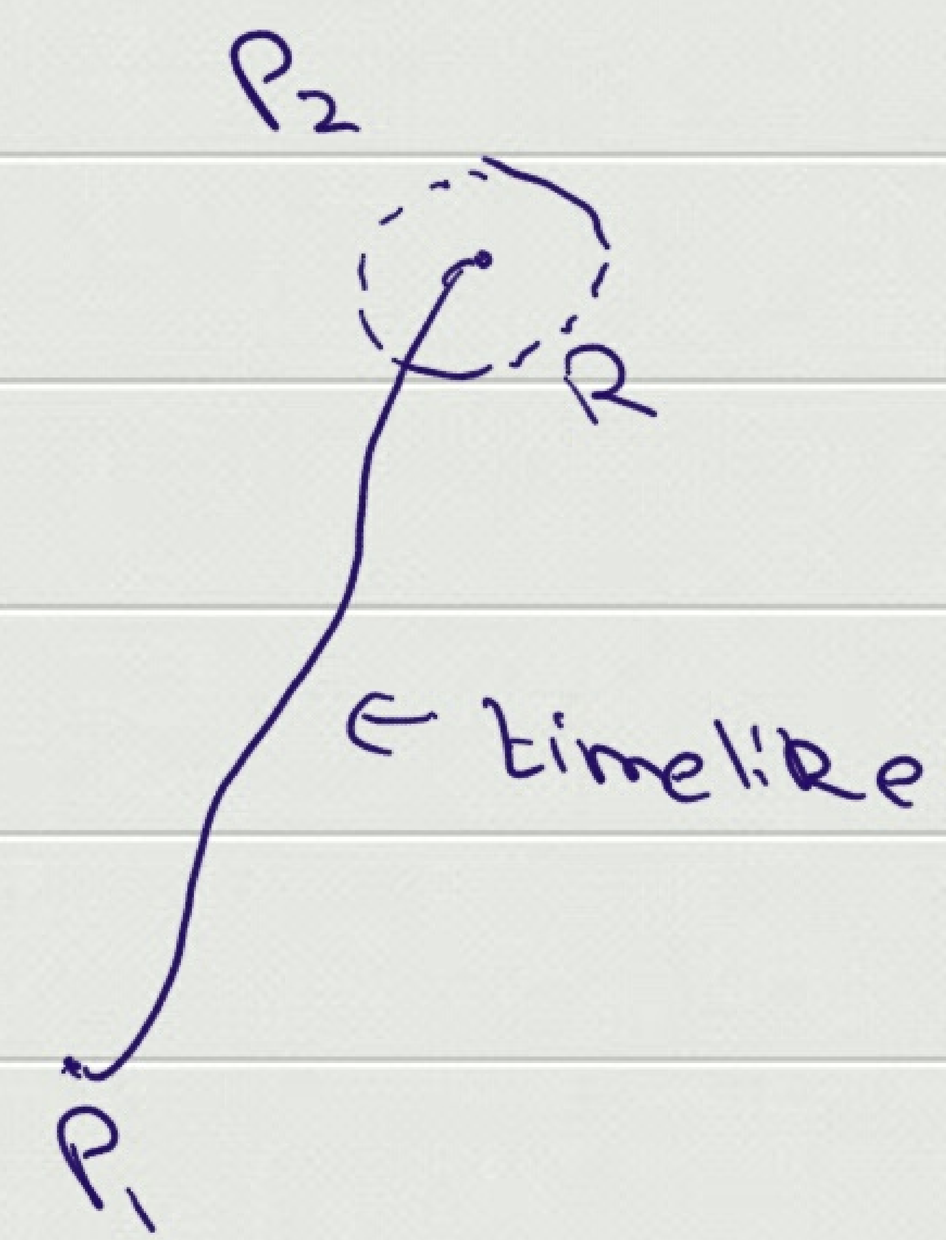



we will now prove that the horizon is made up of null-geodesics that do not intersect. The fact that it is a null hypersurface is also clear from the figure above.

A key intermediate result is that if event $P_1 \in H$ and $P_2 \in H$ then $P_1$ is not in the past of $P_2$.

<u>Proof:</u>

Assume $P_2$ is in the future of $P_1$. then a timelike curve connects $P_1$ to $P_2$

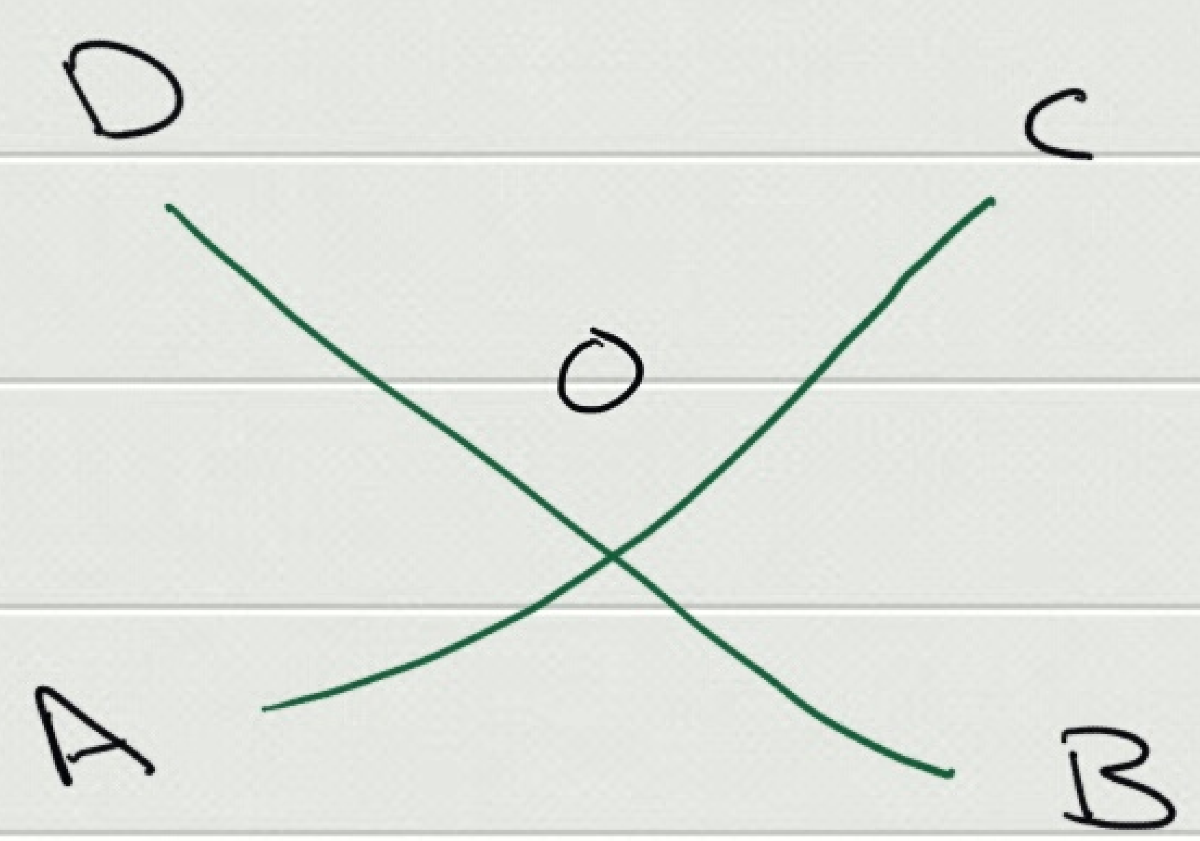


By continuity we have timelike curves connecting $p_1$ to a neighbourhood of $p_2$. But the neighbourhood of $p_2$ intersects the causal past of $\mathscr{I}^+$.
So some timelike curve connects $p_1$ to $\mathscr{I}^+$ which contradicts the assumption that $p_1 \in H$.

Next notice that the horizon cannot be entirely a spacelike surface
This is because there is a pt $p'$ in the neighbourhood of a point $P \in H$, there is a timelike curve connecting $p'$ to $\mathscr{I}^+$. as we move $p'$ to $P$, this curve becomes null

This null curve is on the boundary of the causal past of $\mathcal{I}^+$. So the horizon comprises null curves and spacelike curves.

Next, these null curves cannot intersect. Consider two null curves



If $AOC$ is lightlike and $BOD$ is lightlike then there is a time like curve from $A$ to $D$.

Proof:

Move $ABCD$ infinitesimally close to $O$. $AO$ and $BD$ are future directed null vectors. So $(AO + BD)^2 < 0$

$\Rightarrow AO + BD$ is timelike.

So the null generators of the horizon cannot intersect.

Finally, we prove that the null generators are hypersurface orthogonal.

Say $\xi^\mu$ is a null tangent vector to the horizon.

Now lets take a spacelike cut of the horizon. If $\xi^\mu$ is not orthogonal to this cut, we can write

$$\xi^\mu = h^\mu + o^\mu$$

where $h^\mu$ is along the horizon and $\langle o, h \rangle = 0$.

$h$ must be spacelike since the horizon has only spacelike and null tangents.

But then $o^\mu = \xi^\mu - h^\mu$ is also tangent to the horizon. But so $o^\mu$ must also be spacelike.
But then

$$0 = \xi^2 = (h^\mu + o^\mu)^2 = \langle h, h\rangle + \langle o, o\rangle + 2\underbrace{\langle h, o\rangle}_{0 \text{ by assumption}}$$

$> 0$. !

So the tangent vectors are hypersurface orthogonal.

# Area Theorem

Now we reach the area theorem!. Since the horizon is made of hypersurface orthogonal null geodesics if $\theta < 0$ at any point, the geodesics will either

(a) intersect $\leftarrow$ NOT ALLOWED

or (b) hit a physical singularity $\leftarrow$ NOT ALLOWED BY ASSUMPTION

so $\theta \geq 0 \Rightarrow \frac{dA}{d\lambda} \geq 0$ !

Q.E.D.

We now turn to the $0^{th}$ and $1^{st}$ laws.
The role of temperature is played by the surface gravity.

For stationary black holes, there is a Killing vector field $\chi^a$ that becomes null at the horizon. This vector field is both tangent and normal to the horizon

We already have the tangent + normal vector field that generates the horizon. So we just need to ensure that it is also a Killing vector for the geometry

Let us examine the Schwarzschild black hole. The vector we are looking for is $\frac{\partial}{\partial t}$.

But these coordinates are confusing. In which direction does $\frac{\partial}{\partial t}$ point at the horizon?

with $u = r_* - t$ ; $V = r_* + t$, we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial V} - \frac{\partial}{\partial u}$$

Recall we moved to Kruskal coordinates

$$V_K = e^{\alpha v} \quad ; \quad U_K = -e^{\alpha u}$$

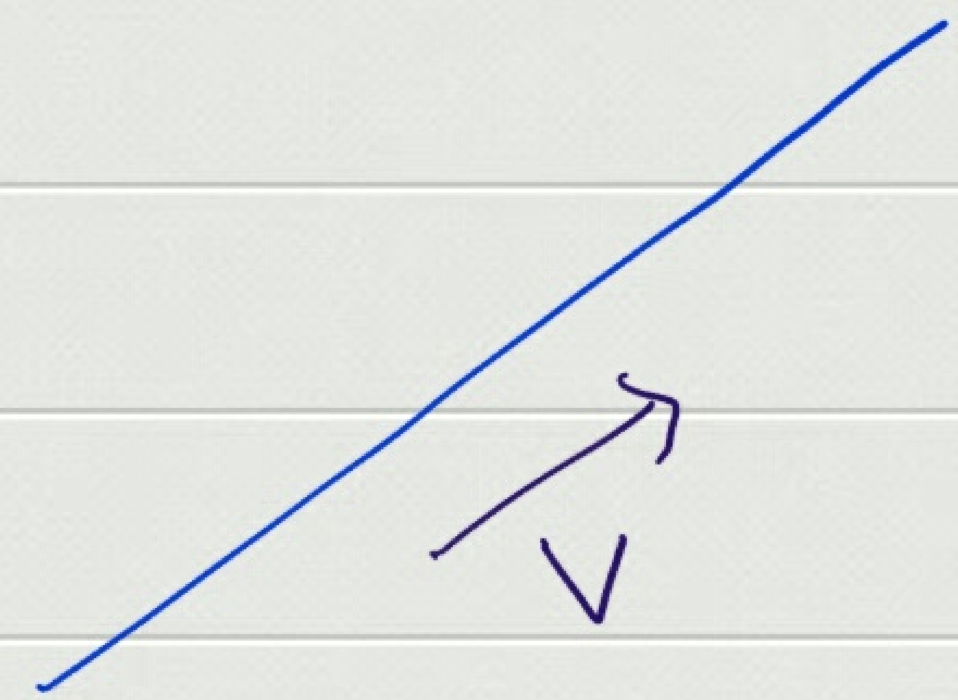
then

$$\frac{\partial}{\partial v} = \alpha V_K \frac{\partial}{\partial V_K} \quad ; \quad \frac{\partial}{\partial u} = \alpha U_K \frac{\partial}{\partial U_K}$$

so $$\frac{\partial}{\partial t} = \alpha V_K \frac{\partial}{\partial V_K} - \alpha U_K \frac{\partial}{\partial U_K}$$

$$\longrightarrow \alpha V_K \frac{\partial}{\partial V_K}$$ at the horizon.

So the picture is as follows




the vector $\chi$ is tangent (and normal!) to the horizon.

For stationary B.H. such a normal killing vector always exists. This is very plausible. But we will not prove the existence of $\chi$ rigorously.

We normalize $\langle \chi, \frac{\partial}{\partial t} \rangle \to -1$ at infinity.

Since $\chi^a \chi_a = 0$, we must have

$$\nabla_\mu (\chi^a \chi_a)$$

is normal to the horizon.

So

$$\boxed{\nabla_\mu (\chi^a \chi_a) = -2\kappa \chi_\mu .}$$

Def$^n$ of $\kappa$ → surface gravity.

We can also write

$$2\chi^a \nabla_\mu \chi_a = -2 \chi^a \nabla_a \chi_\mu$$

Since $\nabla_\mu \chi_a + \nabla_a \chi_\mu = 0$, by Killing's eq$^n$

so

$$\chi^a \nabla_a \chi_\mu = k \chi_\mu$$

If we had parameterized geodesics on the horizon using an affine parameter, we would get 0 on the R.H.S.

$k$ measures the failure of the killing parameter to be an affine parameter. We have

$$\chi^\mu = \frac{dx^\mu}{dv}$$

$dv \leftarrow$ killing param

If we set

$$\frac{d\lambda}{dv} = e^{\kappa v} \quad \text{or} \quad \lambda = \frac{1}{\kappa} e^{\kappa v}$$

then

$$\frac{d x^\mu}{dv} = \frac{d\lambda}{dv} \frac{d}{d\lambda} = e^{\kappa v} \frac{d}{d\lambda} x^\mu$$

$$\Leftarrow \boxed{\chi^\mu = \kappa \lambda k^\mu}$$

so with

$$k^\mu = e^{-\kappa v} \frac{d x^\mu}{dv} = e^{-\kappa v} \chi^\mu$$

we have

$$k^\nu \nabla_\nu k^\mu = e^{-\kappa v} \chi^\nu \nabla_\nu e^{-\kappa v} \chi^\mu$$

$$= e^{-2\kappa v}\left[ \chi^\nu \nabla_\nu \chi^\mu + e^{\kappa v}\left(\chi^\nu \nabla_\nu e^{-\kappa v}\right) \chi^\mu \right]$$

But $\chi^\nu \nabla_\nu e^{-\kappa v} = \dfrac{d e^{-\kappa v}}{dv} = -\kappa\, e^{-\kappa v}$

So

$$k^\mu \nabla_\mu k^\nu = 0 \, !$$

So $\lambda$ is an affine parameter

$$\boxed{\frac{d\lambda}{dV} = e^{-\kappa V}}$$