QABH - Lecture 5. - More on Rindler Space

We already have the important results we need. But, in fact, it is not so hard to work out the full Bogoliubov transformations back to the Minkowski modes.

We already have the

$$\text{Rindler} \longleftrightarrow \text{Unruh}$$

coefficients. So we just need the

$$\text{Unruh} \longleftrightarrow \text{Minkowski}$$

coefficients.

To do this we need to evaluate

$$\int_{-\infty}^{\infty} U^{i\omega/a} \, e^{i\omega' U} \, dU$$

where the branch cut is in the upper half plane and $\omega' > 0$.

For example, we can choose the branch cut so that with $U = |U| e^{i\theta}$, $0 < \theta < \pi$

$$U^{i\omega/a} = \begin{cases} |U|^{i\omega/a} \, e^{-\theta\omega/a}, & \text{for } \theta < \pi/2 \\ |U|^{i\omega/a} \, e^{(2\pi-\theta)\omega/a}, & \text{for } \frac{\pi}{2} < \theta < \pi \end{cases}$$

The initial integration contour is the red-line. We can deform this to the green contour, so the only contribution comes from the branch cut.

So the integral is

$$\int_{-\infty}^{\infty} U^{i\omega/a} e^{i\omega' U} dU = \left(e^{3\pi\omega/2a} - e^{-\pi\omega/2a}\right) \int_{0}^{\infty} |U|^{i\omega/a} e^{-\omega'|U|} \, i \, d|U|$$

$$= i\left(e^{3\pi\omega/2a} - e^{-\pi\omega/2a}\right)\left(\frac{1}{\omega'}\right)^{(i\omega/a+1)} \Gamma\left(\frac{i\omega}{a}+1\right)$$

$$= 2ie^{(\pi\omega/2a)} \left(\frac{1}{\omega'}\right)^{(i\omega/a+1)} \sinh\left(\frac{\pi\omega}{a}\right) \Gamma\left(\frac{i\omega}{a}+1\right).$$

It is now straightforward to write the Rindler modes in terms of the Minkowski modes.

We now turn to a discussion of Rindler space in higher dimensions (>2).
We will use this example to introduce some techniques that will be very useful in the black hole setting.
Although higher dimensional Rindler space can be analyzed exactly, we will take a different approach to analyzing it. At the end of this analysis, we will show that:

"smoothness of the short distance two-point fn $\Rightarrow$ thermal occupancy for Rindler modes."

Ref: see arXiv:1211.6767 Appendix B.

Consider $(d+1)$-dimensional Minkowski space:

$$ds^2 = dt^2 - dz^2 - dx_{d-1}^2$$

We introduce

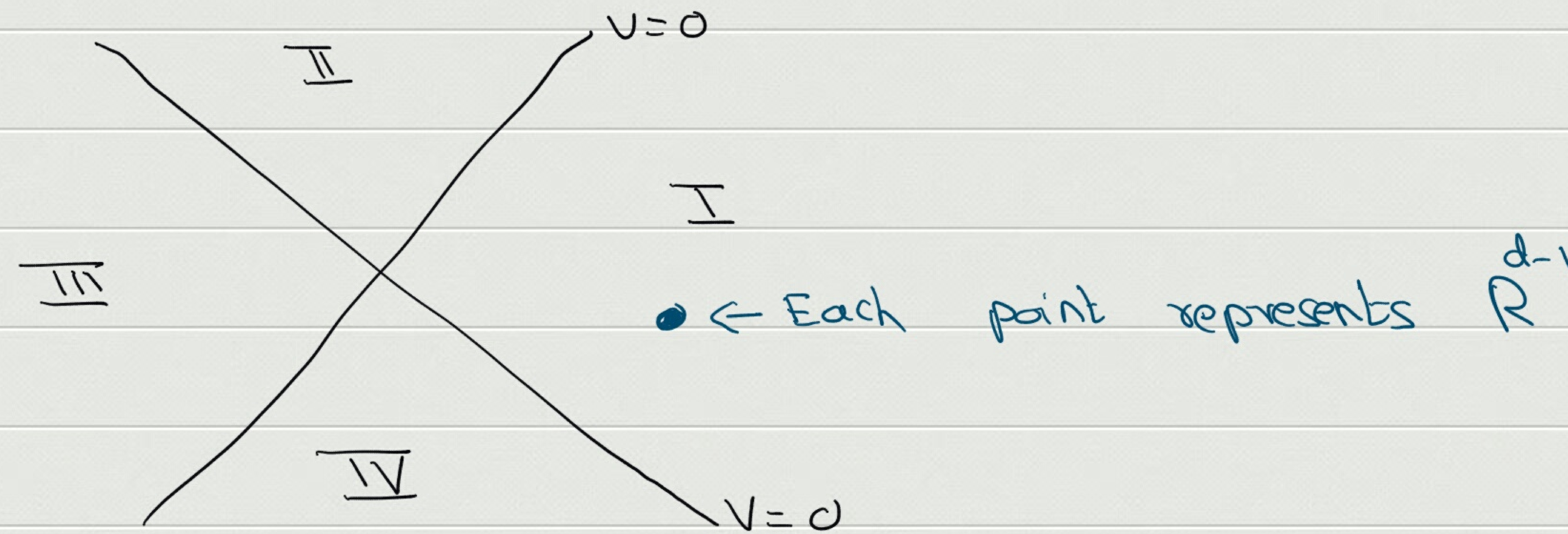
$$U = t - z$$

$$V = t + z$$

so

$$ds^2 = dU dV - dx_{d-1}^2$$

Once again, it is natural to divide spacetime in 4 regions



In I, we introduce

$$U = -e^{-u_R} \qquad \text{[Here, I'm setting } a=1.]$$
$$V = e^{v_R}$$

Now

$$dU\,dV = e^{v_R - u_R}\, du_R\, dv_R - dx_{d-1}^2$$
$$= e^{2x_R}\left(dt_R^2 - dx_R^2\right) - dx_{d-1}^2$$

We have

$$\sqrt{-g} = e^{2x_R} = \frac{1}{g^{t_R t_R}} = \frac{1}{g^{x_R x_R}}$$

So the wave eqn for a minimally coupled massless scalar is:

$$-\omega^2 \phi_R - \partial^2_{x_R}\phi_R + e^{2x_R}\cdot k^2\,\phi_R = 0$$

where

$$\phi = \phi_R(x_R)\, e^{i\omega t_R}\, e^{ik\cdot x_{d-1}}$$

The solution to this is

$$\phi_k = K_{i\omega}\left(|k| e^{x_R}\right)$$

This is a real function; $K_{-i\omega}\left(|k| e^{x_R}\right) = K_{i\omega}\left(|k| e^{x_R}\right)$

It is uniquely picked out by its behaviour as $x_R \to \infty$, where we need the function to remain finite.

So spatial modes here are like standing waves.

We can expand the field as

$$\phi = \int \frac{a_{\omega,k}}{\sqrt{\omega}} e^{-i\omega t_R} e^{i k \cdot x_{d-1}} \frac{K_{i\omega}(|k| e^{x_R})}{|\Gamma(i\omega)|} + h.c.$$

where $|\Gamma(i\omega)|$ is a normalization constant to ensure
$[a_\omega, a^+_{\omega'}] \propto \delta(\omega-\omega')$.

These wave-functions are substantially more complicated than the previous ones

But: near the Rindler horizon, they simplify substantially. There, we have

$$\phi \xrightarrow[U \to 0]{} \int \frac{d\omega}{\sqrt{\omega}} \, a_{\omega, k} \, e^{ik \cdot x_d} \left[e^{-i\omega V_R} e^{-i\delta} + e^{-i\omega U_R} e^{i\delta} \right]$$

with $$e^{i\delta} = \left(\frac{|k|}{2} \right)^{i\omega} \frac{\Gamma(-i\omega)}{|\Gamma(i\omega)|}$$

Moreover, this near horizon expansion is unchanged if we add mass or any other interaction

With an interaction term, the wave equation would become

$$\Box \phi + V'(\phi) = 0$$

But recall that

$$\Box \phi = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{-g}\, \partial_\nu \phi$$

so eqn is

$$\partial_\mu g^{\mu\nu} \sqrt{-g}\, \partial_\nu \phi + \sqrt{-g}\, V'(\phi) = 0.$$

But near the horizon $\sqrt{-g} = e^{+2x_R} \to 0$

so $V(\phi)$ becomes irrelevant!

Now it is possible to play the same game as last time to derive the Bogoliubov coefficients between these modes and the Minkowski modes

But we will instead use invariance of correlators to show that

$$\langle a_\omega a^\dagger_{\omega'} \rangle = \frac{1}{1-e^{-2\pi\omega}} \delta(\omega-\omega').$$

Moreover this behaviour is universal for $\omega \gg 1$ and independent of the details of the coupling.

We will proceed as follows.

1) We will analyze the short distance behavior of the 2-pt function in Minkowski space

2) We will then show that these 2-pt correlators are necessary to reproduce this correlator.

First consider the Minkowski correlator.

$$\langle \phi(U_1, V_1, x_1)\, \phi(U_2, V_2, x_2) \rangle = \frac{1}{(-\delta U \delta V + \delta x^2)^{(d-1)/2}}$$

$$\langle \partial_{v_1} \phi(u_1, v_1, x_1)\ \partial_{v_2} \phi(u_2, v_2, x_2) \rangle$$

$$= -\frac{(d+1)(d-1)}{4} \frac{\delta u^2}{(-\delta u\, \delta v + \delta x^2)^{(d+3)/2}}$$

We now take the points to be almost on the light-cone, so that $\delta u \to 0$, but keep $\delta u\, \delta v < 0$.

In this limit, the 2-pt fn is universal in any theory governed by an asymptotically free fixed point

In this limit, we argue that

constant, fixed below

$$\lim_{\delta u \to 0} \frac{\delta u^2}{(-\delta u \delta v + \delta x^2)^{(d+3)/2}} = \frac{K}{\delta v^2} \delta^{d-1}(\delta x)$$

Note that for $\delta x \neq 0$, $\lim_{\delta u \to 0} \frac{\delta u^2}{(-\delta u \delta v + \delta x^2)^{(d+3)/2}} = 0$

Consider $\int d^{d-1}\delta x \frac{(\delta u)^2}{(-\delta u \delta v + \delta x^2)^{(d+3)/2}}$

rescale $\quad y = \dfrac{\delta x}{\sqrt{-\delta u \, \delta v}}$

then the integral is

$$\int d^{d-1} y \, \frac{\delta u^2}{(y^2+1)^{(d+3)/2}} \, \frac{|\delta u \, \delta v|^{(d-1)/2}}{|\delta u \, \delta v|^{(d+3)/2}}$$

$$= \frac{1}{(\delta v)^2} \int \frac{d^{d-1} y}{(y^2+1)^{(d+3)/2}} = \frac{K}{(\delta v)^2}$$

The final result is

$$\lim_{\delta V \to 0} \langle \partial_{V_1} \phi(U_1, V_1, x_1)\, \partial_{V_2} \phi(U_2, V_2, x_2) \rangle = -k \frac{(d+1)(d-1)}{4} \frac{\delta^{d-1}(\delta x)}{\delta V^2}$$

Physics becomes universal and ultra-local in the light-cone limit!

Next time, we will re-derive this result using Rindler modes.