QABH Lecture 4 - Rindler Space

Last time, we introduced Rindler space, and today we will find the Bogoliubov transformations between the Rindler and the Minkowski quantization.

First, we use a trick to find the relationship between the vacua.
Then we will verify this by finding the full Bogoliubov transformations

Recall that the two expansions we had were

$$\Phi = \int \frac{dw}{\sqrt{w}} \Big[\tilde{a}_w U_L(u_r) + b_w V_L(V_R) + \tilde{a}_w U_R(u_R)$$
$$+ \tilde{b}_w V_R(V_R) + h.c \Big]$$

Rindler Expansion and

On the other hand, we can also write

$$\Phi = \int \frac{dw}{\sqrt{w}} \Big[c_w e^{-iw(t-x)} + d_w e^{-iw(t+x)} + h.c. \Big]$$

Minkowski Expansion

where

$$V_L(u_R) = e^{i\omega u_R}, \quad \text{Region III (on left)}$$
$$= 0, \quad \text{region I (on right)}$$

$$V_R(u_R) = \begin{matrix} 0 \\ e^{-i\omega u_R} \end{matrix} \quad \begin{matrix} \text{, Region III} \\ \text{, Region I} \end{matrix}$$

$$V_L(v_R) = e^{i\omega v_R}, \quad \text{III}$$
$$= 0, \quad \text{I}$$

$$V_R(v_R) = \begin{matrix} 0 \\ e^{-i\omega v_R} \end{matrix} \quad \begin{matrix} \text{III} \\ \text{I} \end{matrix}$$

and the coordinate transformations are:

$$U = -\frac{1}{a} e^{-a u_R}$$

$$V = \frac{1}{a} e^{a v_R}$$

Region $\underline{\mathrm{I}}$

$$U = \frac{1}{a} e^{-a u_R}$$

$$V = \frac{-1}{a} e^{a v_R}$$

Region $\underline{\mathrm{III}}$

Consider the mode

$$U_\omega(u_R) = U_L^*(u_R)\,, \quad \text{in } \mathrm{III}$$

$$= e^{+\pi\omega/a}\, U_R(u_R)\,, \quad \text{in } \mathrm{I}$$

This can be written as

$$U_\omega(u_R) = a^{i\omega/a}\, U^{+i\omega/a}$$

ordinary Minkowski

$U = T - X$

[Imp: recall the minus sign $U \propto e^{-u_R}$]

But in writing this mode, we have to choose the branch cut, which we chosen in the upper-half plane.

Therefore as we continue U from +ve to -ve, we get

$$(-U)^{+i\omega/a} = \left[e^{-\pi i} |U|\right]^{+i\omega/a} = e^{+\pi\omega/a} |U|^{-i\omega/a}$$

The choice of the branch cut controls whether we get $e^{-\pi\omega/a}$ or $e^{\pi\omega/a}$

These modes have the property that

$$\int_{-\infty}^{\infty} U^{+i\omega/a} e^{-i\omega' U} \, dU = 0 \, , \quad \forall \, \omega' > 0$$

because the integral can be continued analytically in the lower half plane.
So, we can write

$$U^{+i\omega/a} = \int_{0}^{\infty} \chi(\nu) e^{-i\nu U} \, d\nu$$

for some $\chi(\nu)$

The Unruh mode has only positive Minkowski frequencies.

We can also consider

$$\tilde{U}_U(u_R) = \begin{cases} U_L(u_R), & \underline{\overline{III}} \\ e^{-\pi w/a} U_R^*(u_R), & \underline{\overline{I}} \end{cases}$$

where again the branch cut was chosen in the upper half plane. and this leads to the -ve sign in the exponent.

Now we can quantize the field as

$$\phi = \int \frac{d\omega}{\sqrt{\omega}} \Big[e_\omega U_\omega(u_R) + \tilde{e}_\omega \tilde{U}_\omega(u_R) + f_\omega V_\omega(v_R) + \tilde{f}_\omega \tilde{V}_\omega(v_R) + h.c. \Big]$$

where V are functions of v defined similarly

[Exercise for the reader!]

We clearly have,

$$a_\omega = e^{+\pi\omega/a} e_\omega + e^{-\pi\omega/a} \tilde{e}^\dagger_\omega$$

$$\tilde{a}_\omega = e^\dagger_\omega + \tilde{e}_\omega$$

$$\tilde{a}_\omega^+ = e_\omega + \tilde{e}_\omega^+$$

So

$$e_\omega = \left(a_\omega - e^{-\pi\omega/a}\tilde{a}_\omega^+\right) / \left[e^{\pi\omega/a} - e^{-\pi\omega/a}\right]$$

Similarly

$$\tilde{e}_\omega = \left(\tilde{a}_\omega - e^{-\pi\omega/a} a_\omega^+\right) / \left[1 - e^{-2\pi\omega/a}\right]$$

[Note the asymmetries in the denominator come because we have not normalized e_ω, $\tilde{e}_\omega$.]

The Minkowski vacuum satisfies

$$e_\omega |\Omega_M\rangle = \tilde{e}_\omega |\Omega_M\rangle = 0$$

and so,

$$(a_\omega - e^{-\pi\omega/a} \tilde{a}_\omega^\dagger) |\Omega_M\rangle = 0$$

$$(\tilde{a}_\omega - e^{-\pi\omega/a} a_\omega^\dagger) |\Omega_M\rangle = 0$$

This can be written as

$$|\Omega_M\rangle = e^{\int d\omega \, e^{-\pi\omega/a} a_\omega^\dagger \tilde{a}_\omega^\dagger + b\text{-terms}} |\Omega_{I,III}\rangle$$

where

$$a_\omega |\Omega_{I,III}\rangle = \tilde{a}_\omega |\Omega_{I,III}\rangle = 0.$$

If we expand this state, we find it has the form [in the Rindler Fock basis]

$$|\Omega_M\rangle = \prod_\omega \sum_{n_\omega} \frac{(a_\omega^+ \tilde{a}_\omega^+)^{n_\omega}}{n_\omega!} e^{-n_\omega \pi \omega/a} |\Omega_{I,III}\rangle$$

$$= \sum_{\{n_\omega\}} e^{-\sum_\omega n_\omega \pi \omega/a} |\{n_\omega\}, \{\tilde{n}_\omega\}\rangle$$

$$= \sum_E e^{-\beta E/2} |E, E\rangle$$

where $\beta = \frac{2\pi}{a}$ and $E = \sum \omega n_\omega$ is the ener of the state.

Such a state is called a thermofield state because tracing over III, we find a mixed state thermal density matrix for I

$$\rho_{\mathrm{I}} = \frac{1}{2} e^{-\beta H}$$

and $\beta = \frac{2\pi}{a}$.

↑

CENTRAL RESULT!

To complete the story, we also write down the expansion in the "Forward" region, region II.
The "Unruh" expansion can easily be extended into region II.
Now we find that

$$e_\omega U_v(u_R) + \tilde{e}_\omega \tilde{U}_v(u_R) + e_\omega^+ U_v^*(u_R) + \tilde{e}_\omega^+ \tilde{U}_v^*(u_R)$$
$$= (e_\omega + \tilde{e}_\omega^+)\, U^{i\omega/a} + (\tilde{e}_\omega + e_\omega^+)\, U^{-i\omega/a}$$
$$= \tilde{a}_\omega^+\, U^{i\omega/a} + \tilde{a}_\omega\, U^{-i\omega/a}$$

Similarly, the V_v and $\tilde{V}_v$ terms add to $b_\omega V^{-i\omega/a} + b_\omega^+ V^{i\omega/a}$.

In region II

$$U = \frac{1}{a} e^{-a u_R} \; ; \; V = \frac{1}{a} e^{a v_R}$$

and so

$$dU\,dV = \frac{1}{a^2} e^{a(v_R - u_R)} \left(-du_R\, dv_R\right)$$

$$= \frac{1}{a^2} e^{2a x_R} \left(dx_R^2 - dt_R^2\right)$$

so x_R is the timelike coordinate and the expansion is

$$\phi_{II} = \int \frac{d\omega}{\sqrt{\omega}} \left[e^{-i\omega v_R}\, b_\omega + e^{i\omega u_R}\, \tilde{a}_\omega + h.c \right]$$

signs correspond to $-i\omega x_R$ since x_R is the time coordinate here

The result to remember is

In region II, by continuity notice that the "v" modes from the right [left movers] and the "u" modes from the left [right movers] cross over.

The right-movers from the right stay there and the left movers from the left also do not cross over.